

DYNAMICS AND SCATTERING OF A MASSLESS PARTICLE

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ABSTRACT.

We estimate the probability of a photon to move faster than light, under the Einstein dynamics which, unlike the wave equation or Maxwell wave dynamics, has singular dispersion relation at zero momentum. We show that this probability goes to zero with time, using propagation estimates suitably multi-scaled to control the contribution of low frequencies. We then prove minimal velocity bounds.

SECTION 0. INTRODUCTION

Let $F(\lambda > c)$ stand for a smoothed out characteristic function of the interval $\lambda > c$. We say that a quantum system, with a configuration coordinate $x \in \mathbb{R}^3$, obeys a maximal velocity bound, if for any solution $\psi(x, t)$, localized in a compact energy interval, there is a positive constant c , s.t.

$$\|F\left(\frac{|x|}{t} > c\right) \psi(x, t)\|_{L^2} \rightarrow 0,$$

as $|t| \rightarrow \infty$.

Here we use the notation (Property F)

$$(F) \quad F(\lambda > c) \equiv \begin{cases} 1 & \lambda > c + \delta \\ 0 & \lambda \leq c - \delta \end{cases}, \quad 0 \leq F, F' \in C^\infty.$$

It is well known that such bounds hold for non-relativistic N -particle quantum systems (see [SS, HSS]). This bounds are still open for massless particles interacting with such systems. See however [Kit, GJY] and the recent works [BFSig,FSig1-2]. The goal of the present paper is to prove such bounds for a model of a single photon (and some simple generalization) with an effective interaction described by a potential $V(x)$.

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New tools will be developed to treat this problem and similar systems. The main difficulty here is the singular nature of the symbol of the corresponding hamiltonian, given by $|\xi|$ near $\xi = 0$. This behavior is typical to systems with massless particles (photons of QED, neutrinos of QCD, quasi-particles in solid state systems, e.g. rotons and more). The breakdown of the standard commutator methods was dealt with in several previous works by the introduction of non self-adjoint multipliers (conjugate operators) and/or mollifying the interaction terms near zero frequencies [BFSig,FSig1-2 and cited references].

The approach developed here, which extends standard positive commutator methods to singular symbols may be of general interest; these methods can be used at the thresholds of the operators in question, where the Mourre theory or propagation estimates fail in general. In particular we derive *quantitative Mourre estimates* at thresholds for certain classes of hamiltonians.

The importance of solving this problem stems from its direct application to various approximate models of QED and other QFT theories with massless particles. As shown very recently [FSig1-2], one key step to proving velocity bounds for photons of non-relativistic type QED models, is a proper construction of a positive commutator at the one particle massless case, which can then be lifted to the Fock space by the standard second quantization formalism. Furthermore, the rigorous analysis of the photo-electric effect, even at the level of finite particle systems is still open[GZ and cited references].

The method we use in this work is to localize dyadically the energy/frequency as we approach the singularity at zero. On each dyadic interval, we can construct propagation observables and get the desired estimates, since the energy is away from zero frequency (by an amount of order 2^{-n} , n large). Furthermore, since we are away from zero the derivatives of the symbol are bounded, though we still lose powers of 2^n . Then we add up the contributions from all dyadic energy intervals, and use the adapted propagation estimates to each such interval. A crucial observation is that by having the initial data localized in space, the contributions from each interval shrink to zero as n goes to infinity, due to (an extension of) Hardy's inequality that controls $E_n(H) < x >^{-1}$ by order 2^{-n} . Here $E_n(H)$ is the projection of the operator H on the dyadic interval $[2^{-n}, 2^{-n+1}]$.

The model we study is given by the time-dependent Schrödinger equation

$$(0.1a) \quad i \frac{\partial \psi}{\partial t} = H\psi \quad \psi|_{t=0} = \psi_0,$$

with the Einstein-Schrödinger self-adjoint operator H on $L^2(\mathbb{R}^3)$ with Domain $H^1(\mathbb{R}^3)$, given by

$$(0.1b) \quad H := (-\Delta)^{1/2} + V(x) \equiv |p| + V(x).$$

This Hamiltonian is a simplified, scalar field version of the restriction of the photon field in QED, to the one particle sector [Ger,GJY]. We think of the equation (0.1) as a model describing dynamics of a single photon. Moreover, it is used as a laboratory for developing the methods to study propagation with singular dispersion at zero energy, needed in tackling the full non-relativistic QED problem. Notice that the equation without the interaction term, is relativistic invariant. Moreover, by multiplying the equation by $-i\partial_t - |p|$, we get the wave equation. However, since the initial data for the derived wave equation, is not localized:

$$\dot{\psi}(0) = |p|\psi(0),$$

we can not use the finite propagation property of the wave equation for this model.

We make the following assumptions:

(i) $V(x)$, $x \cdot \nabla V(x)$, $\langle x \rangle \Delta V(x)$ are sufficiently regular and decay faster than $O(|x|^{-2})$ at infinity;

(ii) H has no zero energy resonances or zero energy eigenvalues.

The main result of this paper is the following theorem:

Theorem. *Let $\psi(x, t)$ be the solution of the Schrödinger equation (0.1), and assume, furthermore, that H satisfies the conditions (i) and (ii). If the initial data is localized in energy in some compact interval, and is such that $\langle x \rangle^{1+\epsilon} \psi_0$ has a bounded L^2 norm, we have the following maximal velocity bound:*

$$(0.2) \quad \left\| F\left(\frac{|x|}{t} > R\right) \psi(x, t) \right\|_{L^2} = o(1) \|\langle x \rangle^{1+\epsilon} \psi_0\|_{L^2}$$

for $R > 1$ and $|t| \rightarrow \infty$. Similar estimates hold for $|x|/t < \theta < 1$.

As was mentioned above, the proof of such a propagation estimates for general N -body Hamiltonians is known, see e.g. [Sig-Sof, HSS]. See also [BFS and cited references]. However, the fact that the one particle Hamiltonian of a massless photon is singular at zero energy, prevents a direct extension of the general theory of propagation estimates to the field, see however [BFSS, FGS,Kit].

The proof is based on the construction of propagation observables, on each (dyadic) scale of the energy. Suppose that the length of the momentum $p = -i\nabla$, $|p|$, is localized in $[2^{-n-1}, 2^{-n}]$, $n \geq 0$. Consider then the operator with F having the property (F):

$$F(A/Rt2^{-n} > 1)$$

defined through the spectral theorem, with A is the self-adjoint generator of dilation

$$(0.4) \quad A = -i/2(x \cdot \nabla + \nabla \cdot x),$$

and $R > 1$.

We start from the basic identity, derived from the Schrödinger equation:

$$(0.5) \quad \partial_t \langle \psi(t), F\psi(t) \rangle = \langle \psi(t), \{i[H, F] + \partial_t F\} \psi(t) \rangle \equiv \langle \psi(t), D_H F \psi(t) \rangle.$$

Here, $F = F(t)$ stands for any family of self-adjoint operators, for which the differentiation can be justified, for the chosen $\psi(t)$. To get a useful propagation estimate, we use the idea of negative propagation observables, [Sig-Sof]: Suppose F stands for a family of time dependent operators, bounded from above (e.g., negative), and such that its Heisenberg derivative, $D_H F$, defined above, is positive up to integrable (in time) corrections. We will then obtain, upon integration over time of (0.5), the following propagation estimate:

$$(0.6) \quad \int_1^T \|B\psi(t)\|^2 dt \leq C \|\psi(0)\|^2.$$

Here, we use that the Heisenberg derivative, $D_H F$, is given by a positive operator, plus integrable corrections:

$$(0.7) \quad D_H F = i[H, F] + \partial_t F = B^* B + R(t), \quad \int_1^\infty \langle \psi(t), R(t) \psi(t) \rangle dt < C \|\psi(0)\|^2.$$

In general, we need to use phase space operators which, while not pseudo-differential, will have suitable phase space support. If we choose for F , the function of A , defined above as our propagation observable, we need to estimate the commutator of H with this F , suitably localized in some energy shell. One can then use the following basic commutator expansion Lemma:

Let H and A be self-adjoint operators on a Hilbert space \mathcal{H} and suppose that H is bounded. To say that the commutator $i[H, A]$ is bounded means that the quadratic form

$$i[(H\psi, A\psi) - (A\psi, H\psi)]$$

on $D(A)$ is bounded and thus defines a bounded, symmetric operator called $i[H, A]$. In the same sense we assume that the higher commutators

$$ad_A^{(k)}(H) = [ad_A^{(k-1)}(H), A]$$

are bounded for $k = 2 \dots n$. Let f be a real C^∞ -function on R . Then, under a further condition given below, the commutator $[H, f(A)]$ has an expansion:

$$(*) \quad [H, f(A)] = \sum_{k=1}^{n-1} \frac{1}{k!} f^{(k)}(A) ad_A^{(K)}(H) + R_n$$

with a remainder estimate

$$\|R_n\| \leq c_n \|ad_A^{(n)}(H)\| \sum_{k=0}^{n+2} \int dx (1 + |x|)^{k-n-1} |f^{(k)}(x)|.$$

The further condition on f is that the integrals above exist. The number c_n is a numerical constant depending on n but not on f, A or H . In particular, the expansion holds if

$$f^{(k)}(x) = O(|x|^{n-\varepsilon-k}) \quad (x \rightarrow \pm\infty)$$

for $k = 1 \dots n+2$, i.e. if the function $f(x)$ grows not faster than $|x|^{n-\varepsilon}$, with corresponding slower growth of the successive derivatives. In that case (*) is defined in form sense on the domain of $f^{(1)}(A)$.

Therefore,

$$(0.8) \quad \iota[H, F(A)] = F'(A)\iota[H, A] + R([H, A], A),$$

where the remainder is given by an explicit multiple integral, involving the group generated by A , and, the double commutator above, of H with A . By the above expansion, we obtain, through symmetrization of the first term on the RHS of (0.8), that the above propagation observable has Heisenberg derivative that is smaller than:

$$(0.9) \quad -\frac{1}{t}G_n^2(A/t) + R(t, n), \quad G_n^2(A, /t) \equiv \lambda F'(\lambda)|_{\lambda=A/Rt2^{-n}}.$$

So, G_n stands for a bump function of $\frac{A}{Rt2^{-n}}$ around 1. We will get a useful estimate, if we can control the remainder term, $R(t, n)$, by an integrable function of t , which is also well behaved as n tends to infinity. The leading term in the expression (0.9), comes from the time derivative of F , of the Heisenberg derivative. The first term in the Heisenberg derivative, the commutator, has an expansion beginning with:

$$(0.10) \quad F'_n(A/R2^{-n}t)\iota[H, A]\frac{2^n}{Rt} = F'_n(A/R2^{-n}t)\iota[|p| - 2x \cdot \nabla V]\frac{2^n}{Rt},$$

which can be shown to be much smaller than the leading term, by using the energy localization around 2^{-n} . To see that, we use that the commutator $\iota[H, A]$ is bounded by $c|p|$, for some finite constant c , by the use of the uncertainty principle:

$$\begin{aligned} & |(\psi, i[H, A]\psi)| \\ &= |(\psi, (|p| - 2x \cdot \nabla V)\psi)| \\ &\leq (\psi, |p|\psi) \\ &+ c(\psi, \langle x \rangle^{-2-\varepsilon} \psi) \\ &\leq c(\psi, |p|\psi). \end{aligned}$$

and our decay and regularity assumptions on V . Then, we need to show that energy localization implies a similar momentum localization. This can not be done using standard localization arguments, since the derivative of the localization function grows like $2^n!$. To this end, a completely different argument is used: One proves that under generic spectral assumptions on H , $HP_c(H)$ dominates a constant times $P_c(H)|p|P_c(H)$. $P_c(H)$ stands for the spectral projection of H on its continuous spectral part.

It then follows that:

$$(0.11) \quad E_n(H)|p|E_n(H) \leq c2^{-n}E_n(H)^2.$$

Finally, we need to show that the remainder $R(t, n)$, higher commutator terms in the expansion of the Heisenberg derivative, are integrable in t . This is the most involved step; without energy localization the formal expression for the remainder is given by a divergent integral. The way to estimate this last remainder term, is to use the fact that the group generated by dilations moves the support of the energy localization functions. Consequently, the integrals over the group actions are limited to finite domains. The remainder term comes from the following expansion:

$$(0.12) \quad \begin{aligned} E_n i[|p|, \Phi_n] E_n &= E_n i[|p|^{1/2}, \Phi_n] |p|^{1/2} E_n + E_n |p|^{1/2} i[|p|^{1/2}, \Phi_n] E_n \\ &= E_n |p|^{1/2} F'_n(A/t) |p|^{1/2} E_n \frac{1}{Rt} + 2\Re E_n |p|^{1/2} R_2(A/t, n) E_n, \end{aligned}$$

with

$$\Phi_n \equiv E_n F\left(\frac{A}{tR2^{-n}} > 1\right) E_n$$

and with $R_2(A/t, n)$ given by

$$(0.13) \quad \begin{aligned} R_2(A/t, n) &= \frac{1}{2} \int d\lambda \hat{F}_n(\lambda) e^{i\lambda A/Rt} \int_0^\lambda ds \int_0^s du e^{-iuA/Rt} \frac{1}{2} |p|^{1/2} e^{iuA/Rt} (Rt)^{-2} \\ &= \frac{1}{4} \int d\lambda \hat{F}_n(\lambda) e^{i\lambda A/Rt} \int_0^\lambda ds \int_0^s du e^{-u/Rt} |p|^{1/2} (Rt)^{-2}. \end{aligned}$$

If we try to bound the expression for R_2 by taking the norm of the first term on the RHS of (0.13), as it was done in past works, we lose a factor of $2^{-n/2}$, coming from localizing the $|p|^{1/2}$ factor. On the other hand, we can not directly estimate the last expression on the RHS of equation (0.13), since the integrand grows exponentially in λ , while \hat{F}_n decays slower than exponential, being the Fourier transform of a

compactly supported function. To this end, we use the fact that the dilation group, generated by A , changes the support of functions of $|p|$, or H :

$$(0.14) \quad E_{I_n}(|p|)e^{i\lambda A}E_{I_n}(|p|) = E_{I_n}(|p|)E_{I_n}(|e^{-\lambda}p|)e^{i\lambda A} = 0,$$

for $|\lambda| \geq \ln 2$. Then, we use the mutual domination of $|p|, H$:

$$(0.15) \quad P_c(H)H \leq cP_c(H)|p|P_c(H) \leq dP_c(H)H,$$

for some positive constants c, d .

Combining equations (0.14), (0.15), we can then show that the integration on λ is limited to a compact domain in λ , in equation (0.13). Collecting all of the above, (0.5 - 0.15), we get estimates of the form:

$$(0.16) \quad E_n(H)\frac{1}{t}G_n^2(A/t)E_n(H) \in L^1(dt).$$

This estimate is then jacked up by the use of the propagation observable $\frac{A}{t}F_n(A/t)$, to obtain,

$$(0.17) \quad E_n(H)F_n(A/t)E_n(H)\frac{1}{t} \in L^1(dt).$$

In the next step of the proof, we estimate using the above, the following operator:

$$E_n(H)F\left(\frac{|x|}{t} > R\right)E_n(H).$$

We write: $F\left(\frac{|x|}{t} > R\right) = F\left(\frac{|x|}{t} > R\right)F_n(A/t) + F\left(\frac{|x|}{t} > R\right)\bar{F}_n(A/t)$.

$$F_n + \bar{F}_n = 1.$$

The first term of the above decomposition, goes to zero, as time goes to infinity, by the above propagation estimates, on $F_n(A/t)$.

So, we need to show that the \bar{F}_n term also goes to zero.

This is formally true, since it consists of a product of two operators, with disjoint classical phase-space support, on the energy shell 2^{-n} .

Again, the proof of this property necessitates the use of new phase space localization arguments. In the final step of the proof, we sum over all n , and in the process we lose some powers of 2^{-n} . These are compensated by requiring the initial data to be localized in x , and by using that (up to 2) negative powers of $|p|$, are bounded, up to a constant, by positive powers of $|x|$.

There are new difficulties in completing this argument, compared with the usual case, without dyadic energy localization.

First, we need to minimize the number of powers of 2^n , coming from expanding the function F_n . Then, we need to trade positive powers of the momentum (derivative operator) $p = -i\nabla$, for powers of 2^{-n} .

Finally, to control the remainder term in the Commutator Expansion Lemma, the Q_2 term, (or R_2 term), we need to commute the derivative through the dilation group, which produces exponentially large factors.

The way out of these problems involves the following arguments. To limit the integrations in the remainder term R_2 , we notice that the dilation group moves the dyadic energy interval, away from its original support. Hence, for large enough value of the group parameter, λ , the fact that our propagation observable is localized on the dyadic interval, from both sides, gives an extra decay, that cancels the exponential growth factor. This is shown in detail in the subsection "The term R_2 ".

To get the 2^{-n} factor from the momentum p , we prove some propositions about the properties of the operator H , which might be of independent interest. (see Proposition (2.7)) Specifically, we show, that in three (and higher) dimensions, if there are no zero energy resonances and eigenvalues, then H and $|p|$ dominate each other, up to a multiplicative constant, on the continuous spectral subspace of H .

These estimates are the key to getting the right minimal powers of 2^n , from the various propagation estimates and phase space localizations.

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SECTION 1. PROPAGATION ESTIMATES

Our goal in this section is to prove the following key propagation estimate, as sketched in the introduction:

$$(1.1a) \quad \int_1^T \|F_n(A/t)E_n\psi(t)\|^2 \frac{dt}{t} \leq C \|E_n\psi(0)\|^2.$$

and with

$$(1.1b) \quad F_n(A/t) \equiv F\left(\frac{A}{Rt2^{-n}} > 1\right).$$

E_n stands for the operator $E_{I_n}(H)$. We use propagation estimates, with the propagation observables

$$(1.2a) \quad \Phi_n \equiv E_{I_n}(H) F \left(\frac{A}{tR2^{-n}} > 1 \right) E_{I_n}(H),$$

$$(1.2b) \quad \Phi_n^{(1)} \equiv E_{I_n}(H) \frac{A}{t} F \left(\frac{A}{tR2^{-n}} > 1 \right) E_{I_n}(H),$$

where, $n = 0, 1, \dots, \infty$, I_n stands for the interval $[2^{-n-1}, 2^{-n}]$, E_{I_n} is the characteristic function of I_n and A is the dilation generator,

$$(1.3) \quad A \equiv \frac{1}{2}(x \cdot p + p \cdot x).$$

On the support of F , at least classically, $|x| \geq \delta t$, since p can not be unbounded on the support of $E_n(H)$. Consequently, we can drop, to leading order the contribution of $V(x)$ to H . So, in fact, to leading order, $E_n(H) \approx E_n(|p|)$. Therefore at the classical level, $\frac{|x|}{t} \geq R$ on the support of Φ_n , and $R \leq \frac{|x|}{t} \leq 2R$ on the support of $E_n F' E_n$.

Furthermore,

$$(1.4) \quad |\partial_\lambda^j F \left(\frac{\lambda}{Rt2^{-n}} > 1 \right)| \sim \left(\frac{2^n}{Rt} \right)^j.$$

The main propagation estimate is based on showing that:

$$(1.5a) \quad \begin{aligned} & E_{I_n} i \left[|p| + V, F \left(\frac{A}{Rt2^{-n}} > 1 \right) \right] E_{I_n} \\ &= E_n \left\{ i|p|^{1/2} \left[|p|^{1/2}, F \right] + i \left[|p|^{1/2}, F \right] |p|^{1/2} + i[V, F] \right\} E_n \\ &= E_n \left\{ |p|^{1/2} \tilde{F}^2 |p|^{1/2} + |p|^{1/2} \frac{2^{2n}}{R^2 t^2} Q_2(A/t) |p|^{1/2} \right\} E_n + O(t^{-2}), \end{aligned}$$

where $\tilde{F}^2 = \frac{2^n}{Rt} F'$ and with the bounds: $E_n Q_2(A/t) E_n = O(1)$ for $t \geq 2^n/R$, and of order 2^{-n} , for $t \leq 2^n/R$. Then, we show that

$$(1.5b) \quad E_n |p|^{1/2} \tilde{F}^2 |p|^{1/2} E_n \lesssim (c/Rt) E_n G_n^2 E_n,$$

with

$$G_n^2 = F \left(\frac{A}{Rt2^{-n}} \sim 1 \right).$$

Letting $F_n(\lambda) = F(2^n \lambda \geq 1)$ and using that $-\frac{A}{t}F'_n \sim -R2^{-n}F'_n, R \gg 1$, we obtain, using also

$$(1.6) \quad \int_1^T \partial_t \langle \psi_t, E_{I_n}(H) F_n\left(\frac{A}{Rt}\right) E_{I_n}(H) \psi_t \rangle dt = \int_1^T \langle \psi_t, \iota[H, \Phi_n] \psi_t \rangle dt \\ + 2^n R^{-1} \int_1^T \langle \psi_t, E_{I_n}(H) \left(\frac{-A}{t^2}\right) F'_n E_{I_n}(H) \psi_t \rangle dt.$$

that bounds (1.5), (1.6) imply:

Theorem 1.1. *Under the previous assumptions on H , and assume that (1.5) holds, we have the following propagation estimates:*

$$(1.7a) \quad \int_1^T \|F'_n(A/Rt) E_n \psi(t)\|^2 \frac{dt}{t} \leq c(R) \|E_n \psi(0)\|^2$$

$$(1.7b) \quad \int_1^T \|G_n(A/Rt) E_n \psi(t)\|^2 \frac{dt}{t} + \langle \psi(T), E_n \frac{A}{T} F_n(A/RT) E_n \psi(T) \rangle \\ \leq c(R) \| \langle A \rangle^{1/2} E_n \psi(0) \|^2$$

Proof.

To prove (1.7a), we note that it is the same as (0.6) with

$$B = \tilde{F}_n(A/t) E_n(H).$$

As explained in the introduction, (0.6) follows upon integration over time, from 0 to T of the identity (0.5) and application of the fundamental theorem of calculus, provided that we prove the following commutator estimate:

$$(1.8) \quad D_H \Phi_n = D_H E_n(H) F_n\left(\frac{A}{tR} > 1\right) E_n(H) \\ = -\frac{\tilde{c}(R)}{t} E_n(H) \tilde{F}_n^2(A/t) E_n(H) + O(L^1(dt)).$$

With this choice of Φ_n , the integral over time of (0.5) takes the form (1.6).

Using the definition that

$$(1.9) \quad G_n^2(A/t) \equiv 2^n \frac{A}{Rt} F'_n(A/t),$$

the second term on the RHS of (1.6) is seen to be equal to

$$(1.10) \quad - \int_1^T \langle \psi(t), E_n(H) \frac{1}{t} G_n^2(A/t) E_n(H) \psi(t) \rangle dt$$

which corresponds to the first term on the RHS of (1.8). Therefore, the main estimate is to show that the correction terms are in $L^1(dt)$, uniformly in 2^n . This is achieved by proving Reverse Mourre Estimate [Sig-Sof]:

$$(1.11) \quad \|E_n i[H, A] E_n\| \lesssim 2^{-n}.$$

We also need to control from above the second order correction term in the expansion of the commutator $i[H, F_n(A/t)]$ by the commutator expansion Lemma.

These are proved in Proposition A, B, together with the general inequalities of Section 2. To proceed, note that the commutator in the correction term in (1.6) - the first term, is expanded in (1.5a) as a sum of three terms: \tilde{F}^2 term, Q_2 term and $O(t^{-2})$. The first term, the \tilde{F}^2 term, is dominated (for $R > c > 1$) by a small constant times the leading term in 1.8, since

$$\tilde{F}^2 = \frac{2^n}{Rt} F'_n(A/t) = \frac{1}{R} G_n^2(A/t)$$

and $E_n(H)|p|^{1/2} = O(2^{-n/2})$, (Section 2, by Prop 2.7). Next, we consider the remainder terms Q_2 and $O(t^{-2})$. These terms come from the expansion of the commutator

$$(1.12) \quad E_n i[H, F_n(A/t)] E_n = E_n i[V, F_n(A/t)] E_n + E_n i[|p|, F_n(A/t)] E_n.$$

We proceed by proving some Preliminary Estimates.

Lemma 1.2. *For $W(x)$, smooth and decaying fast enough,*

$$(1.13) \quad B_n \equiv \|F_n(A/t) W(x) E_n(H)\| \lesssim \frac{1}{Rt}$$

Proof of Lemma.

$$(1.14) \quad \begin{aligned} B_n &= \frac{1}{2} \|F_n(A/t) A^{-1} (x \cdot p + p \cdot x) W(x) E_n(H)\| \\ &= O(2^n/Rt) \{ \|(xW' + W + xW') E_n(H)\| \\ &\quad + 2\|xW(x)p E_n(H)\| + \|W(x) E_n(H)\| \} \\ &= O(2^n/Rt) \{ \|\langle x \rangle^{-1} E_n(H)\| \| |xW| + |x^2 W'| \|_{L^\infty} \} \\ &\leq \left(\frac{1}{Rt} \right). \end{aligned}$$

We used that $A^m F_n(A/t) = O([2^{-n}/Rt]^m) F_n(A/t)$, and $\|pE_n(H)\| + \|\langle x \rangle^{-1} E_n(H)\| = O(2^{-n})$. which proves the Lemma. \blacksquare

Therefore, using $F_n(A/t)A^{-2} = \tilde{F}_n(A/t)2^{2n}/t^2$ we get from the above computation that

$$(1.15) \quad \begin{aligned} \|F_n(A/t)W(x)E_n(H)\| &\lesssim (2^{2n}/R^2t^2) \times \\ &\|\langle x \rangle^{-1} E_n(H)\| \|\langle x \rangle^2 |W| + \langle x \rangle^3 (|W'| + |W''|)\|_{L^\infty} \\ &\leq c(2^n/R^2t^2). \end{aligned}$$

From this we immediately conclude that

$$(1.16) \quad \int_{\varepsilon 2^n}^{\infty} dt \langle \psi(t), E_n(H) i[V, F_n(A/t)] E_n(H) \psi(t) \rangle \leq \frac{1}{\varepsilon R^2} \|V\| \|E_n(H)\psi\|^2.$$

The shorttime estimate, $t \leq 2^n/R$, follows from the bound:

$$(1.17) \quad \begin{aligned} &|\langle E_n\psi(t), (HF_n - F_nH)E_n(H)\psi(t) \rangle| \\ &\leq 2\|HE_n\psi(t)\| \|F_n\| \|E_n\psi(t)\| \\ &\leq 2 \cdot 2^{-n} \|E_n(H)\psi(t)\|^2. \end{aligned}$$

$$(1.18) \quad \begin{aligned} &\Rightarrow \left| \int_0^{2^n/R} \langle E_n(H)\psi(t), i[H, F_n(A/t)] E_n(H)\psi(t) \rangle dt \right| \\ &\leq C \|E_n(H)\psi(t)\|^2 / R. \end{aligned}$$

Next, we consider the second term on the RHS of (1.12):

$$(1.19) \quad E_n i[|p|, F_n(A/t)] E_n$$

The estimate (1.18) applies to the left side of eq. (1.12), so, we only need to estimate the above commutator for $Rt > 2^n$.

The above commutator can be written as (1.5a), after using that

$$(1.20) \quad i[|p|, F_n] = i|p|^{1/2} [|p|^{1/2}, F_n] + i[|p|^{1/2}, F_n] |p|^{1/2}$$

and then expanding the commutators, using the commutator expansion Lemma:

$$(1.21) \quad \begin{aligned} i[|p|^{1/2}, F_n] &= F'_n i \left[|p|^{1/2}, \frac{A}{Rt2^{-n}} \right] + R_2 \\ &= (Rt)^{-1} 2^n F'_n(A/t) \left(+\frac{1}{2} |p|^{1/2} \right) + R_2. \end{aligned}$$

Since $F' \geq 0$, we can write

$$(1.22) \quad \tilde{F}^2 \equiv \frac{2^n}{Rt} F',$$

Furthermore, by taking the adjoint of eq. (1.21) we get

$$i[|p|^{1/2}, F_n] = \frac{1}{2}|p|^{1/2} \tilde{F}^2 + \bar{R}_2.$$

(1.20 - 1.22) implies eq. (1.5a). For R large enough, the first term on the RHS of (1.5a), while positive, is dominated by the leading term (1.10). So it remains to control the integral over time of the R_2 terms, only for $t > 2^n/R$. For $t \leq 2^n/R$ we bound the whole commutator $i[H, F_n]$, see eq. 1.18.

This is shown in propositions A, B. We now consider the proof of (1.7b). To this end, we employ a similar proof as for (1.7a), but with a new propagation observable

$$(1.23) \quad \Phi_n^{(1)} \equiv \frac{A}{t} F_n(A/t).$$

The left hand side is now an **unbounded** operator, but nonnegative, since $F_n(x)$ is supported in $x \geq 2^{-n}R$. Hence, the argument leading to (1.18), for $t \leq 2^n/R$ does not apply. The proof in this case is done in the next section, proposition D. The RHS consists, as in eq. (1.6) from a commutator term and a derivative w.r.t time. The derivative w.r.t. time gives

$$(1.24) \quad -\frac{A}{t^2} F_n(A/t) - \frac{A}{t^2} \frac{2^n}{R} F'_n(A/t) \leq 0.$$

The expression in (1.24) is easily seen to be less than $-\frac{A}{t^2} F_n(A/t) - \frac{1}{t} \tilde{F}^2(A/t) \leq -\frac{R}{t} F_n$ with

$$(1.25) \quad \tilde{F}^2(x) = F\left(\frac{2^n x}{R} = 1\right),$$

that is, a C_0^∞ bump function around $R2^{-n}$.

The commutator term will be shown to be dominated by

$$(1.26) \quad \frac{2^{-n}}{t} F_n(A/t) + \frac{1}{Rt} \tilde{F}^2(A/t)$$

in proposition C.

Hence (1.26) is dominated by (1.25) for $R > 1$. The higher order terms R_2 , are controlled in proposition C. Upon integration overtime, the bound (1.7b) follows. \blacksquare

We now proceed to prove the statements that were assumed above. This is the content of the next 4 propositions A-D.

We begin with estimating the $E_n(H)R_2(H, F_n(A))E_n(H)$ for $t > R^{-1}2^n$.

Proposition A (The $R_2(H, F_n)$ term).

For Φ_n as in (1.2a), we have: (Q_2 is defined via equation (1.5a))

$$(1.27) \quad \int_{2^n/R}^T \langle E_n | p |^{1/2} Q_2 | p |^{1/2} E_n \rangle dt \leq \frac{c}{R} \|E_n \psi(0)\|^2.$$

Proof.

Direct application of the commutator expansion lemma gives:

$$(1.28) \quad \begin{aligned} i \left[|p|^{1/2}, F_n(A/t) \right] &= \int d\lambda \hat{F}_n(\lambda) e^{i\lambda A/Rt} \frac{1}{Rt} \int_0^\lambda ds e^{-isA/Rt} i \left[|p|^{1/2}, A \right] e^{isA/Rt} \\ &= \frac{1}{2} F'_n(A/t) \frac{1}{Rt} |p|^{1/2} + \tilde{R}_2(A/t) |p|^{1/2} = \frac{1}{2} F'_n(A/t) \frac{1}{Rt} |p|^{1/2} + R_2(A/t), \end{aligned}$$

where we used that

$$(1.29) \quad -i \left[A, |p|^{1/2} \right] = p \cdot \nabla_p |p|^{1/2} = \frac{1}{2} |p|^{1/2}.$$

$$(1.30) \quad \begin{aligned} R_2(A/t) &= -\frac{1}{2} \int d\lambda \hat{F}_n(\lambda) e^{i\lambda A/Rt} \int_0^\lambda ds \int_0^s du \quad e^{-iuA/Rt} \frac{1}{2} |p|^{1/2} \quad e^{iuA/Rt} (Rt)^{-2} \\ &= \frac{1}{4} \int d\lambda \hat{F}_n(\lambda) e^{i\lambda A/Rt} \int_0^\lambda ds \int_0^s du \quad e^{+u/Rt} |p|^{1/2} (Rt)^{-2}. \end{aligned}$$

We used $e^{i\alpha A} |p| e^{-i\alpha A} = e^{-\alpha} |p|$. In general, this integral blows up at infinity, due to the fact that $e^{u/Rt}$ grows exponentially fast, while $\hat{F}_n(\lambda)$ decays faster than any polynomial, but not exponentially, since $\hat{F}_n(\lambda)$ is the Fourier transform of a compactly supported function. Applying energy localization by $E_n \equiv E_n(H)$

$$(1.31) \quad \begin{aligned} E_n i \left[|p|, \Phi_n \right] E_n &= E_n i \left[|p|^{1/2}, \Phi_n \right] |p|^{1/2} E_n + E_n |p|^{1/2} i \left[|p|^{1/2}, \Phi_n \right] E_n \\ &= E_n |p|^{1/2} F'_n(A/t) |p|^{1/2} E_n \frac{1}{Rt} + E_n |p|^{1/2} 2 \operatorname{Re} R_2(A/t) E_n \\ &\equiv E_n \frac{|p|^{1/2}}{Rt} F'_n(A/t) |p|^{1/2} E_n \\ &\quad + E_n |p|^{1/2} Q_2 |p|^{1/2} E_n \end{aligned}$$

with $R_2(A/t)$ given by (1.30). Let $\tilde{E}_n(\cdot)E_n(\cdot) = E_n(\cdot)$.

$$(1.32) \quad E_n(H) = E_n(H) - E_n(|p|) + E_n(|p|) \equiv \delta E_n + E_n(|p|).$$

Hence, the $R_2(A/t)$ term becomes

$$(1.33) \quad \begin{aligned} E_n(H)|p|^{1/2}R_2(A/t)E_n(H) &= |p|^{1/2}E_n(|p|)R_2(A/t)E_n(|p|) \\ &\quad + \delta E_n|p|^{1/2}R_2(A/t)E_n(|p|) \\ &\quad + |p|^{1/2}E_n(|p|)R_2(A/t)\delta E_n \\ &\quad + \delta E_n|p|^{1/2}R_2(A/t)\delta E_n \\ &\equiv J_1 + J_2 + J_3 + J_4. \end{aligned}$$

$$\bar{E}_n \equiv 1 - \tilde{E}_n.$$

In our case, $1 - \tilde{E}_n = \tilde{E}(|p| \leq 1) - \tilde{E}_n$. \tilde{E}_n stands for smoothed E_n function of $|p|$, and where we used proposition (2.4d) to get that $\bar{E}_n(|p|)E_n(H) = \bar{E}_n(|p|)O(n2^{-n})E_n(H)$.

The estimate of J4 follows from that of J2, J3, as it is higher order by a factor of $n2^{-n}$.

$$(1.34) \quad \begin{aligned} \int_1^T J_1 dt &= \int_1^T |p|^{1/2}E_n(|p|)R_2(A/t)E_n(|p|)dt \\ &= \int_1^T \frac{dt}{R^2 t^2} |p|^{1/2}E_n(|p|)Q_2(A/t)E_n(|p|)|p|^{1/2} \\ &= \int_1^{c2^n} \frac{dt}{R^2 t^2} |p|^{1/2}E_n(|p|)Q_2(A/t)E_n(|p|)|p|^{1/2} \\ &\quad + \int_{c2^n}^T \frac{dt}{R^2 t^2} |p|^{1/2}E_n(|p|)Q_2(A/t)E_n(|p|)|p|^{1/2}. \end{aligned}$$

If $T \leq c2^n$, then the second term on the r.h.s of (1.34) is zero.

We now estimate the the LHS of equation (1.34). Using proposition (2.4c), it follows that the λ integration (and therefore the other integrations) in the expression for $R_2(A/t)$, eq. (1.30), is limited to

$$(1.35) \quad |\lambda| \leq Rt \ln 2.$$

Hence,

$$(1.36) \quad J1 = \frac{c}{(Rt)^2} |p|^{1/2} E_n(|p|) \int_{|\lambda| \leq Rt \ln 2} d\lambda \hat{F}_n(\lambda) e^{i\lambda A/Rt} \\ \times \int_0^\lambda ds \int_0^s du e^{-iuA/Rt} |p|^{1/2} e^{+iuA/Rt} E_n(|p|).$$

$$J1 = \frac{c}{(Rt)^2} |p|^{1/2} E_n(|p|) \int_{|\lambda| \leq Rt \ln 2} \hat{F}_n(\lambda) e^{i\lambda A/Rt} \int_0^\lambda ds \int_0^s du e^{+u/Rt} |p|^{1/2} E_n(|p|) \\ = \frac{O(1)}{(Rt)^2} E_n(|p|) 2^{-n/2} \left(\int_{|\lambda| \leq Rt \ln 2} |\lambda^2 \hat{F}_n(\lambda)| d\lambda \right) 2^{-n/2} E_n(|p|) \\ = \frac{O(1)}{(Rt)^2} E_n(|p|) 2^{-n} (Rt)^2 E_n(|p|) \text{ since } \int |\hat{F}(\lambda)| d\lambda = O(1).$$

Hence,

$$\int_1^{2^n/R} J1 \, dt \leq O\left(\frac{1}{R}\right).$$

If $Rt > 2^n$, we use instead, that

$$\int |\lambda^2 \hat{F}_n(\lambda)| \, d\lambda \leq 2^{2n},$$

so that,

$$J1 = \frac{O(1)}{(Rt)^2} E_n 2^n E_n, \quad \text{and then,}$$

$$(1.37) \quad \int_{2^n/R}^T J1 dt = O\left(\frac{1}{R}\right).$$

next, we estimate the the terms $J2 - J4$ defined in equation (1.33). Consider the region $Rt > 2^n$. The integrand to estimate can be written as

$$(1.38) \quad (Rt)^{-2} \hat{F}_n(\lambda) \tilde{J}(\lambda, s, u),$$

$$(1.39) \quad \tilde{J}(\lambda, s, u) \equiv |p|^{1/2} E_n(|p|) e^{i\lambda A/Rt} |p|_u^{1/2} \delta E_n,$$

$$(1.40) \quad \left(|p|_s \equiv e^{-iAs/Rt} |p| e^{iAs/Rt} \right),$$

and adjoint of such term.

First, we decompose the region of integration λ to:

$$\frac{|\lambda|}{Rt} > m \text{ and } \frac{|\lambda|}{Rt} \leq m, \quad m > \ln 2.$$

For $\frac{|\lambda|}{Rt} > m$, we consider the case $\lambda > 0, \lambda < 0$ separately.

For $\lambda < 0$, we have $(e^{-iuA/Rt} |p|^{1/2} e^{+iuA/Rt} = e^{u/2Rt})$ since $\lambda < 0$ implies $\lambda \leq u, s \leq 0$,

$$(1.41) \quad \begin{aligned} \|\tilde{J}_2(\lambda, s, u)\| &= \| |p|^{1/2} E_n(|p|) e^{i\lambda A/Rt} e^{u/2Rt} |p|^{1/2} \delta E_n \| = \\ &= \| O(2^{-n/2}) \| |p|^{1/2} (E_n(|p|) - E_n(H)) \| = O(2^{-n}), \end{aligned}$$

and where we used that $e^{i\lambda A/Rt}$ is bounded on L^2 , and $e^{u/2Rt} \leq 1$, for $u \leq 0$.

So, the integral of this part ($\lambda < 0$) in J_1 is bounded by

$$(1.42) \quad \begin{aligned} &\int_{2^n/R}^T \frac{dt}{(Rt)^2} \int_{\frac{\lambda}{Rt} \leq -m} |\hat{F}_n(\lambda)| \lambda^2 O(2^{-n}) d\lambda \\ &\leq 2^{-n} R^{-1} O(2^{-n}) \cdot 2^{2n} O(|m|^{-k}) = O(|m|^{-k}) \end{aligned}$$

where we use that

$$(1.43) \quad \int_{2^n/R}^T \frac{dt}{R^2 t^2} \leq c 2^{-n} R^{-1},$$

$$(1.44) \quad \int_{|\lambda| \geq (2^{+n}/R)m} |\hat{F}_n(\lambda)| \lambda^2 d\lambda \leq cR/m.$$

Next, we consider $\lambda \geq 0, \lambda/Rt > m$.

In this case, $0 \leq u \leq \lambda$, and therefore $e^{u/2Rt}$ is large.

So, in this case we commute $e^{i\lambda A/Rt}$ to the right

$$(1.45) \quad \begin{aligned} \|\tilde{J}_2(\lambda, s, u)\| &= \| |p|^{1/2} E_n(|p|) e^{(u-\lambda)/2Rt} |p|^{1/2} e^{i\lambda A/Rt} \delta E_n \| = \\ &= O(2^{-n}). \end{aligned}$$

The estimate of this part of J_2 is therefore the same as the bound (1.41).

The estimate of \hat{J}_3 is identical.

Next, we estimate J_4 :

(1.46)

$$\begin{aligned} J_4 &= \delta E_n |p|^{1/2} R_2(A/t) \delta E_n \\ &= \frac{c}{(Rt)^2} \delta E_n |p|^{1/2} \int_{|\lambda| \geq mRt} d\lambda \hat{F}_n(\lambda) e^{i\lambda A/Rt} \int_0^\lambda ds \int_0^s du |p|^{1/2} e^{iuA/Rt} \delta E_n. \end{aligned}$$

The estimate on J_4 follows from a sharper bound of δE_n , given in the following Lemma:

Lemma 1.3.

Let $E_n(\lambda)$ be as above, a smooth characteristic function of I_n .

$$\text{supp } E_n(\lambda) = [2^{-n-1}(1-\delta), 2^{-n}(1+\delta)].$$

Then

$$(1.47) \quad \|\tilde{E}_n \delta E_n\| = \|\tilde{E}_n(H)(E_n(H) - E_n(|p|))\| \leq C 2^{-n} \|V\|_2.$$

$$(1.48) \quad \| |p|^{\frac{1}{2}+\varepsilon} (\delta E_n) \tilde{E}_n(H) \| \leq C_\varepsilon 2^{-n-\beta n} \|V\|_{\frac{3}{2}+\beta-\varepsilon};$$

$$(1.49) \quad \| |p|^{1/2} \delta E_n \tilde{E}_n(H) \| \leq C_\varepsilon 2^{-n+\varepsilon n-\beta n} \|V\|_{\frac{3}{2}+\beta-\varepsilon},$$

with

$$\|V\|_\alpha \equiv \|\langle x \rangle^\alpha V(x)\|_{L^\infty}.$$

Proof.

(1.50)

$$\begin{aligned} \|\tilde{E}_n \delta E_n\| &= \|\delta E_n \tilde{E}_n\| = \\ &= \left\| \int \hat{E}_n(\lambda) e^{i\lambda|p|} \int_0^\lambda e^{-is|p|} V(x) \langle x \rangle^1 \langle x \rangle^{-1} |p|^{-1} |p| \tilde{E}_n(H) e^{isH} ds d\lambda \right\| = \\ &= \left\| \int \hat{E}_n(\lambda) \left(\partial_\lambda e^{i\lambda|p|} \right) \int_0^\lambda e^{-is|p|} \frac{1}{|p|} V(x) \langle x \rangle^1 \langle x \rangle^{-1} |p|^{-1} |p| \tilde{E}_n(H) e^{isH} ds d\lambda \right\| \\ &\leq \left\| \int \frac{\partial \hat{E}_n(\lambda)}{\partial \lambda} \int_0^\lambda e^{-is|p|} \frac{1}{|p|} \frac{1}{\langle x \rangle} (\langle x \rangle^2 V(x)) (\langle x \rangle^{-1} |p|^{-1}) |p| \tilde{E}_n(H) e^{isH} ds d\lambda \right\| \\ &+ \left\| \int \hat{E}_n(\lambda) \frac{1}{|p|} \frac{1}{\langle x \rangle} (\langle x \rangle^2 V(x)) (\langle x \rangle^{-1} |p|^{-1}) |p| \tilde{E}_n(H) d\lambda \right\|. \end{aligned}$$

The second term on the *RHS* of (1.50) is zero, since

$$\int_{-\infty}^{\infty} \hat{E}_n(\lambda) d\lambda = E_n(0) = 0.$$

The first term on the *RHS* of (1.50) is bounded by

$$\begin{aligned} & \int \left| \lambda \frac{\partial \hat{E}_n(\lambda)}{\partial \lambda} \right| d\lambda \left\| \frac{1}{|p|} \frac{1}{\langle x \rangle} \right\|^2 \|\langle x \rangle^2 V(x)\|_{L^\infty} \| |p| \tilde{E}_n(H) \| \\ & \leq C 2^{-n} \int \left| \lambda \frac{\partial \hat{E}_n(\lambda)}{\partial \lambda} \right| d\lambda \leq C 2^{-n}, \end{aligned}$$

since $\lambda \frac{\partial \hat{E}_n(\lambda)}{\partial \lambda} = \mathcal{F}(xE'_n(x))$ and $xE'_n(x)$ is a C_0^∞ function bounded by $O(1)$.

This proves (1.47).

The proof of (1.48) follows a similar argument:

For $0 \leq \beta \leq 1$:

$$\begin{aligned} & \| |p|^{1/2+\varepsilon} (\delta E_n) E_n(H) \| = \\ & \left\| \int \hat{E}_n(\lambda) \left(\partial_\lambda^2 e^{i\lambda|p|} \right) \right. \\ & \left. \int_0^\lambda e^{-is|p|} |p|^{-3/2+\varepsilon} \langle x \rangle^{-3/2+\varepsilon} \langle x \rangle^{3/2-\varepsilon} V(x) \langle x \rangle^{+\beta} \langle x \rangle^{-\beta} |p|^{-\beta} |p|^\beta \tilde{E}_n(H) e^{isH} ds d\lambda \right\| \\ & \leq c \int \left| \lambda \frac{\partial^2 \hat{E}_n(\lambda)}{\partial \lambda} \right| d\lambda \| V(x) \langle x \rangle^{3/2+\beta-\varepsilon} \| 2^{-\beta n} 2^{-n} + \\ & \quad + c \int \left| \frac{\partial \hat{E}_n(\lambda)}{\partial \lambda} \right| d\lambda 2^{-\beta n} \| \langle x \rangle^{3/2+\beta-\varepsilon} V(x) \|_{L^\infty} \\ & + c \left\| \int \hat{E}_n(\lambda) |p|^{-3/2+\varepsilon} \langle x \rangle^{-3/2+\varepsilon} \langle x \rangle^{3/2-\varepsilon} V(x) \langle x \rangle^\beta \langle x \rangle^{-\beta} |p|^\beta |p|^{-\beta} \tilde{E}_n(H) H e^{i\lambda H} d\lambda \right\| \\ & \leq C 2^{-2n-\beta n} \| V \langle x \rangle^{3/2+\beta-\varepsilon} \|_{L^\infty} + (C 2^{-n} 2^{-\beta n} + C 2^{-n-\beta n}) \| V(x) \langle x \rangle^{3/2+\beta+\varepsilon} \|_{L^\infty}. \end{aligned}$$

Estimate (1.49) follows by interpolation of (1.47) and (1.48).

■

Next, we consider the region $\frac{|\lambda|}{Rt} \leq m$. Now, $\frac{|\lambda|}{Rt} \leq m$ implies $\frac{|s|}{Rt}, \frac{|u|}{Rt} \leq m$. Therefore $e^{|s|/Rt}, e^{|u|/Rt} \leq e^m$, so, $|p|_s \leq e^m |p|$. So, we pick up a factor of 2^{-n} from $|p|^{1/2}$ factors, and the integration of u, s, t gives a quantity

$$(1.51) \quad \int_{\frac{|\lambda|}{Rt} \leq m} |\lambda^2 \hat{F}_n(\lambda)| d\lambda \leq c 2^{2n}.$$

Hence,

$$\left| \int_{2^n/R}^T \frac{dt}{R^2 t^2} \int \int \tilde{J}(\lambda, s, u) \hat{F}_n(\lambda) d\lambda ds du \right| \leq c \frac{e^m/2}{R} 2^{-n} (2^{2n} 2^{-n/2} 2^{-n(3/2-\varepsilon)})$$

Here $2^{-n}/R$ comes from the t integration of $(Rt)^2$. A factor of 2^{2n} comes from the λ integration of equation (1.51). $e^{m/2}$ is the bound on $e^{u/2Rt}$. $2^{-n/2}$ comes from $|p|^{1/2} E_n(H)$. $2^{-n(3/2-\varepsilon)}$ comes from $|p|^{1/2} \delta E_n$. ■

The region $0 \leq t \leq 2^n/R$.

Proposition B.

For Φ_n as in (1.2a), we have:

$$(1.52) \quad \int_1^{2^n/R} \|G_n(A/t) E_n \psi(t)\|^2 \frac{dt}{t} \leq c \|E_n \psi(0)\|^2.$$

Here, G_n , is a bump function of A/t around $2^{-n}/R$.

Proof.

Now we have,

$$\begin{aligned} \frac{d}{dt} (\psi(t), \Phi_n \psi(t)) &= (\psi(t), i[H, \Phi_n] \psi(t)) + \left(\psi(t), \frac{d\Phi_n}{dt} \psi(t) \right), \\ \left(\psi(t), \frac{d\Phi_n(t)}{dt} \psi(t) \right) &= \left(E_n \psi(t), t^{-1} F'_n(A/t) \left(\frac{-2^n}{R} \frac{A}{t} \right) E_n \psi(t) \right) \\ &\leq -\frac{1}{t} \left(E_n \psi, F'_n(A/t) E_n \psi \right) \equiv -\frac{1}{t} \left(E_n \psi, \tilde{F}_n^2(A/t) E_n \psi \right), \end{aligned}$$

where,

$$\begin{aligned} |\tilde{F}_n^2(A/t)| &\lesssim 1 \quad \text{and} \quad \tilde{F}_n^2 \geq 0 \\ \tilde{F}_n^2(A/t) &\simeq 1 \quad \text{for} \quad \frac{A}{t} \sim R 2^{-n} \\ \tilde{F}_n^2(A/t) &= 0 \quad \text{for} \quad \frac{A}{t} \approx R 2^{-n}. \end{aligned}$$

$$\begin{aligned} |(\psi(t), i[H, \Phi_n] \psi(t))| &= |(\psi(t), i(H E_n F_n E_n - E_n F_n E_n H) \psi(t))| \\ &= |(E_n \psi(t), (H E_n F_n E_n - E_n F_n E_n H) E_n \psi(t))| \\ &\leq 2 \cdot 2^{-n} \|E_n \psi(t)\|^2, \end{aligned}$$

since $\|HE_n\| \leq 2^{-n}$, and F_n is bounded. Hence,

$$\begin{aligned} & \int_1^{2^n/R} \left\{ \left(\psi(t), i[H, \Phi_n]\psi(t) \right) + \left(\psi(t), \frac{d\Phi_n}{dt}\psi(t) \right) \right\} dt \\ & \leq - \int_1^{2^n/R} \frac{dt}{t} \left\| \tilde{F}_n(A/t) E_n \psi(t) \right\|^2 + 2 \left\| E_n \psi(t) \right\|^2 \int_1^{2^n/R} 2^{-n} dt \\ & = - \int_1^{2^n/R} \frac{dt}{t} \left\| \tilde{F}_n(A/t) E_n \psi(t) \right\|^2 + \frac{2}{R} \left\| E_n \psi(t) \right\|^2. \end{aligned}$$

Therefore,

$$\begin{aligned} & \int_1^{2^n/R} \frac{dt}{t} \left\| \tilde{F}_n(A/t) E_n \psi(t) \right\|^2 + \left(\psi(2^n/R), E_n F(A \geq 1) E_n \psi(2^n/R) \right) \\ & \leq \left(E_n \psi(1), F_n(A, t=1) E_n \psi(1) \right) + \frac{2}{R} \left\| E_n \psi(t) \right\|^2. \end{aligned}$$

since $F_n(A, t=2^n/R) = F(\frac{A}{2^{-n}(2^n/R)} \geq R) = F(AR \geq R) = (A \geq 1)$.

Improved Decay

Proposition C.

Under the assumptions as the above Propositions (A,B), we have the following propagation estimate,

$$\begin{aligned} & \int_{2^n/R}^T \left\| \frac{A}{t} \right\|^{1/2} F_n \left(\frac{A}{Rt2^{-n}} > 1 \right) E_n \psi(t) \right\|^2 \frac{dt}{t} \\ (1.53) \quad & + \langle \psi(T), E_n \frac{A}{T} F_n \left(\frac{A}{RT2^{-n}} > 1 \right) E_n \psi(T) \rangle \leq c(R) \|E_n |A|^{1/2} F(A \geq R2^{-n}) E_n \psi(0)\|^2 \blacksquare \end{aligned}$$

for all $R \geq 1 + \varepsilon$.

Proof. We can use $E_n \frac{A}{t} F_n(A/t) E_n = \Phi_n$. The estimate of $R_2(A/t)$ is done as before, so for $t \geq 2^n$, it is done as before. The main change is that now $\hat{F}(\lambda)$ is replaced by $\partial_\lambda \hat{F}(\lambda)$. The first term in the commutator expansion of $i[H, \Phi_n]$, has an extra, positive term, which is however integrable over time by the previous propositions (A,B), since it is supported (in phase space) on the support of F'_n .

$$\begin{aligned} & i[H, E_n \frac{A}{t} F_n(A/t) E_n] + E_n \partial_t \frac{A}{t} F_n(A/t) E_n = \\ (1.54) \quad & = E_n \left(-\frac{A}{t^2} F_n - \frac{A^2}{t^3} F'_n(A/t) \right) E_n + E_n i[V(x), \frac{A}{t} F_n(A/t)] E_n \\ & + E_n i[|p|, \frac{A}{t}] F_n(A/t) E_n + E_n \frac{A}{t} i[|p|, F_n(A/t)] E_n \end{aligned}$$

$$(1.55) \quad \leq -\frac{1}{t} E_n \{ 2^{-n} R - |p| + x \cdot \nabla V \} F_n(A/t) E_n$$

$$(1.56) \quad + \frac{1}{t} E_n \left\{ \frac{A}{t} F'_n(A/t) [|p| - x \cdot \nabla V] + \frac{A}{t} R_2(A/t) \right\} E_n$$

$$(1.57) \quad - E_n \frac{1}{t} 2^{-n} R \left(\frac{A}{t} F'_n \right) E_n.$$

Now,

$$(1.58) \quad E_n(-|p| + x \cdot \nabla V) = E_n \tilde{E}_n(H)(-|p| + x \cdot \nabla V) = E_n(2^{-n} + 0(1)2^{-n})$$

$$(1.59) \quad = -\frac{1}{t} E_n \{ 2^{-n} (R - 1 - 0(1)) \} F_n(A/t) E_n.$$

$$(1.60) \quad \leq E_n(2^{-n} R) \frac{1}{t} \tilde{F}_n E_n = 2^{-n} R O(L^1(dt))$$

by our previous propagation estimates.

Since $\frac{A}{t} F'_n(A/t) = \tilde{F}_n(A/t)$, and $(|p| - x \cdot \nabla V) E_n = O(2^{-n})$ it follows that

$$(1.61) \quad O(2^{-n}) E_n \frac{1}{t} \tilde{F}_n(A/t) E_n + E_n \frac{A}{t} R_2(A/t) E_n = O(2^{-n}) O(L^1(dt)) + E_n \frac{A}{t} R_2(A/t) E_n.$$

The term $\frac{A}{t} R_2(A/t)$ is the same as the $R_2(A/t)$ we treated before, except, that through integration by parts, $\frac{A}{t}$ factor is absorbed into the $R_2(A/t)$ term, by replacing $\hat{F}_n(\lambda)$ by $\partial_\lambda \hat{F}_n(\lambda)$.

Since $\partial_\lambda \hat{F}_n(\lambda) = \mathcal{F}(x F_n(x))$, and $x F_n(x) = O(2^{-n} R)$ we conclude that this last term is also of order

$$2^{-n} R O(L^1(dt)).$$

collecting all the estimates on (1.54), we arrive at (1.53), which proves Proposition C.

□

Remark.

Since

$$\begin{aligned} & |(E_n \psi(1), A E_n \psi(1))| \leq C \| |p| E_n H H^{-1} \psi(1) \| [\| [x, E_n] \psi(1) \| + \| E_n x \psi(1) \|] \\ & \leq C 2^{-2n} \| H^{-1} P_c(H) \langle x \rangle^{-1} \| \| \langle x \rangle \psi(1) \| [O(1) \| \psi \| + \| x \psi(1) \|] = O(2^{-2n}) \| \langle x \rangle \psi(1) \|_2^2 \end{aligned}$$

then, by factoring out a factor of $2^{-n} R$, we get a bound on

$$\begin{aligned} \int_1^T \| F_n(A, t) E_n \psi(t) \|^2 \frac{dt}{t} & \leq C (\| E_n \psi \|^2 + 2^{-2n} \| \psi \|_1^2) \\ \| \psi \|_1 & = \| \langle x \rangle \psi(1) \|_{L^2}. \end{aligned}$$

For $0 \leq t \leq 2^n$, since $\frac{A}{t} F_n$ is not bounded, the proof is different.

Proposition D.

$$\int_1^{2^n/R} \|\sqrt{(A/t)} F_n \left(\frac{A}{Rt2^{-n}} > 1 \right) E_n \psi(t)\|^2 \frac{dt}{t} \quad (1.62)$$

$$+ \langle \psi(2^n/R), E_n \frac{A}{2^n/R} F_n (A > 1) E_n \psi(t) \rangle \leq c(R) \|E_n < A^{1/2} E_n \psi(1)\|^2,$$

for all $R > 1 + \varepsilon$.

Proof. $\partial_t \left(\frac{A}{t} F_n \right) = -\frac{A}{t^2} F_n - \frac{A}{Rt^2 2^{-n}} F'_n \leq 0$. This gives the leading term, in the Heisenberg derivative of the Propagation observable $\frac{A}{t} F_n(A/t)$.

We write,

$$(1.63) \quad \frac{A}{t} F_n = F_{n,1} + F_{n,2},$$

where,

$$(1.64) \quad F_{n,1} = i \int e^{-\lambda^2} (\partial_\lambda \hat{F}_n(\lambda)) e^{-i(A/t)\lambda} d\lambda,$$

and

$$(1.65) \quad F_{n,2} \equiv (A/t) F_n - F_{n,1}.$$

Then, $F_{n,2}$ is bounded, and the previous proof applies, while $F_{n,1} \equiv \frac{A}{t} G_n$, with G_n smooth, approaching a constant at infinity. Then,

$$\begin{aligned} i[H, F_{n,1}] &= \frac{1}{t} i[H, A] G_n + \frac{A}{t} i[H, G_n] \\ (1.66) \quad &= \frac{1}{t} [|p| - x \cdot \nabla V] G_n + \frac{A}{t} \int \hat{G}_n(\lambda) e^{-i\lambda A/t} \int_0^\lambda e^{isA/t} [H, A/t] e^{-isA/t} ds d\lambda. \blacksquare \end{aligned}$$

The first term is bounded by $2^{-n} \|G_n\|/t$ on support of E_n , and therefore its integral over $1 \leq t \leq 2^n$ is bounded by $O(1)$. The second term is,

$$\begin{aligned} &\frac{1}{t} \int \partial_\lambda \left(\lambda \hat{G}_n(\lambda) \right) e^{-i\lambda A/t} \frac{1}{\lambda} \int_0^\lambda e^{-s/t} [|p| + \tilde{V}(x, s)] ds d\lambda \\ (1.67) \quad &- \frac{1}{t} \int \hat{G}_n(\lambda) e^{-i\lambda A/t} \frac{1}{\lambda} \int_0^\lambda \left[e^{-s/t} |p| + V(xe^{s/t}) \right] ds d\lambda. \end{aligned}$$

$$\frac{1}{\lambda} \int_0^\lambda e^{-s/t} ds = -\frac{t}{\lambda} e^{-s/t} \Big|_0^\lambda = \frac{t}{\lambda} (e^{-\lambda/t} - 1) = -1 + \frac{1}{2} \left(\frac{\lambda}{t} \right) + O \left(\left(\frac{\lambda}{t} \right)^2 \right)$$

for $\frac{\lambda}{t} \ll 1$.

So the first term of (1.66) contributes $|\int \hat{G}_n(\lambda) e^{i\lambda A/t} d\lambda| \frac{1}{t} 2^{-n} = O(\frac{1}{t}) 2^{-n}$, and the second term is bounded by

$$\int |\partial_\lambda(\lambda \hat{G}_n(\lambda))| d\lambda/t + t^{-1} \int_{|\lambda| \geq 1} |\hat{G}_n(\lambda)| e^{-\lambda/t} / d\lambda$$

We use

$$\int |\hat{G}_n(\lambda)| d\lambda \leq \int |\hat{F}_n(\lambda) e^{-\lambda^2}| + \text{Erf}(\lambda |\partial_\lambda \hat{F}_n(\lambda)|) d\lambda = O(2^{-n});$$

Erf stands for the Error function. Furthermore, there is a factor of 2^{-n} , coming from $|p| + V$: Here $\tilde{V}(x, s) = e^{isA/t} \tilde{V}(x) e^{-isA/t}$. $\tilde{V}(x, s) = \tilde{V}(e^{-s/t} x)$, and for $|\frac{s}{t}| \leq 1$, we use $|p| + \tilde{V}(x, s) \leq C|p|$.

■

SECTION 2. AUXILIARY IDENTITIES AND INEQUALITIES

Lemma 2.1. *Assume $H = |p| + V$ and $|V| < \frac{1}{2r}$. Then,*

$$(2.1) \quad \left\| |p| E_{I_n}(H) f \right\| + \left\| V E_{I_n}(H) f \right\| \leq c 2^{-n} \left\| E_{I_n}(H) f \right\|.$$

Proof. Let $\|f\| = 1$.

$$\begin{aligned} \left(f, E_{I_n}(H) (|p|^2 + V^2) E_{I_n}(H) f \right) &= \left(f, E_{I_n}(H) \{H^2 - V|p| - |p|V\} E_{I_n}(H) f \right) \\ &\leq 2^{-2n} \left\| E_{I_n}(H) f \right\|^2 + 2 \left\| V E_{I_n}(H) f \right\| \left\| |p| E_{I_n}(H) f \right\|. \blacksquare \end{aligned}$$

Let $a = \left\| |p| E_{I_n}(H) f \right\|$ $b = \left\| V E_{I_n}(H) f \right\|$. Then, $b < (1 - \delta) \left\| |p| E_{I_n}(H) f \right\|$ by the uncertainty inequality (in 3-dimensions or higher), and so,

$$\begin{aligned} \left(f, E_{I_n}(H) (|p|^2 + V^2) E_{I_n}(H) f \right) &= (a^2 + b^2 - 2ab) + 2ab \\ &\geq \delta^2 \left\| |p| E_{I_n}(H) f \right\|^2 + 2ab. \end{aligned}$$

It follows that, $\delta^2 \left\| |p| E_{I_n}(H) f \right\|^2 \leq 2^{-2n} \left\| E_{I_n}(H) f \right\|^2$. ■

Proposition 2.2. *If $|p| \lesssim H$, then (2.1) holds.*

Proof. When $|p| \leq mH$, we have that, by the spectral theorem,

$$(2.2) \quad \frac{1}{H} \leq \frac{m}{|p|} \implies \left\| \frac{1}{H^{1/2}} f \right\|_{L^2} \leq m^{1/2} \left\| \frac{1}{|p|^{1/2}} f \right\|.$$

Hence,

$$\begin{aligned} V(x) \frac{1}{H} &= V(x) \left(\frac{1}{|H|} - \frac{1}{|p|} \right) + V(x) \frac{1}{|p|}. \\ \left\| V(x) \frac{1}{|p|} \right\| &= \left\| V(x) r \frac{1}{r} \frac{1}{|p|} \right\| \leq \left\| V(x) \frac{1}{r} \right\|_{\infty} 2 \left\| |p| \frac{1}{|p|} \right\| < \infty, \\ V(x) \left(\frac{1}{H} - \frac{1}{|p|} \right) &= -V(x) \frac{1}{H} V(x) \frac{1}{|p|} \\ &= -V(x) \frac{1}{H} \frac{1}{\sqrt{r}} \sqrt{r} V(x) r \frac{1}{r} \frac{1}{|p|} \\ &= -V(x) \sqrt{r} \frac{1}{\sqrt{r}} \frac{1}{H} \frac{1}{\sqrt{r}} r^{3/2} V(x) \frac{1}{r} \frac{1}{|p|}. \end{aligned}$$

Since,

$$\frac{1}{\sqrt{r}} \frac{1}{H} \frac{1}{\sqrt{r}} = \frac{1}{\sqrt{r}} \frac{1}{H^{1/2}} \frac{1}{H^{1/2}} \frac{1}{\sqrt{r}} = \left(\frac{1}{H^{1/2}} \frac{1}{\sqrt{r}} \right)^* \left(\frac{1}{H^{1/2}} \frac{1}{\sqrt{r}} \right),$$

we get that,

$$\begin{aligned} \left\| V(x) \left(\frac{1}{H} - \frac{1}{|p|} \right) \right\| &\leq \left\| \frac{1}{H^{1/2}} \frac{1}{\sqrt{r}} \sqrt{r} V(x) \right\| \left\| \frac{1}{H^{1/2}} \frac{1}{\sqrt{r}} r^{3/2} V(x) \frac{1}{r} \frac{1}{|p|} \right\| \\ &\leq m \left\| \frac{1}{|p|^{1/2}} \frac{1}{\sqrt{r}} (\sqrt{r} V(x)) \right\| \left\| \frac{1}{|p|^{1/2}} \frac{1}{\sqrt{r}} (r^{3/2} V) \right\| \left\| \frac{1}{r} \frac{1}{|p|} \right\| \\ &< m C_V, \end{aligned}$$

where C_V is a constant, depending on the L^∞ norm of $V, \sqrt{r}V, r^{3/2}V$.

We therefore conclude that,

$$|p| \frac{1}{H} = (|p| + V - V) \frac{1}{H} = H \frac{1}{H} - V \frac{1}{H} = 1 - V \frac{1}{H}$$

is also bounded. Finally, we have that,

$$|p| E_{I_n}(H) = |p| \frac{1}{H} H E_{I_n}(H) = O(1) 2^{-n} E_{I_n}(H). \quad \blacksquare$$

Now we prove some useful identities.

Lemma 2.3. *Let the dimension be 3, and denote as before by A the dilation generator, and by L^2 the Laplacian on the unit sphere. Then the following identities and an inequality hold:*

- (i) $r^2(-\Delta) = A^2 + L^2 - 2iA - 3/4$, L^2 stands for the Laplacian on the unit sphere.
- (ii) $e^{i\lambda A}|p|^\alpha e^{-i\lambda A} = e^{-\alpha\lambda}|p|^\alpha$; $A \equiv \frac{1}{i}(r\partial_r + \frac{3}{2}) = (x \cdot p + p \cdot x)/2$
- (iii) $e^{i\lambda A}r^\alpha e^{-i\lambda A} = e^{\alpha\lambda}r^\alpha$.
- (iv) $\frac{1}{|p|}A\frac{1}{r} = \frac{1}{|p|}\left(i|p|\frac{\partial}{\partial|p|} - \frac{3i}{2}\right)\frac{1}{r} = \frac{1}{|p|}\left(ip \cdot \frac{\partial}{\partial p} - \frac{3i}{2}\right)\frac{1}{r} = O(1)$.
- (v) $\left\|\frac{1}{|p|}\frac{1}{r}\psi\right\| \leq 2\|\psi\|$.

Proof.

- (i) $r^2(-\Delta) = r^2\left(-\partial_r^2 - \frac{2}{r}\partial_r + \frac{L^2}{r^2}\right) = -r^2\partial_r^2 - 2r\partial_r + L^2$
 $= -2r\partial_r + 2r\partial_r - r\partial_r^2 r + L^2 = -r\partial_r^2 r + L^2$,
 $A^2 = -\frac{1}{4}\left(+2r\partial_r + 3\right)^2 = (-4r\partial_r r\partial_r - 9 - 12r\partial_r)\frac{1}{4}$
 $= r\partial_r - r\partial_r^2 r - \frac{9}{4} - 3r\partial_r = -r\partial_r^2 r - 2r\partial_r - \frac{9}{4}$;
 so, $r^2(-\Delta) = -r\partial_r^2 r + L^2 = A^2 + \frac{9}{4} + L^2 + 2iA - 3/4 = A^2 + L^2 - 2iA - 3$.
- (ii) $\partial_\lambda\left(e^{i\lambda A}|p|^\alpha e^{-i\lambda A}\right) = e^{i\lambda A}i\left[A, |p|^\alpha\right]e^{-i\lambda A} = -\alpha e^{i\lambda A}|p|^\alpha e^{-i\lambda A}$
 $\Rightarrow e^{i\lambda A}|p|^\alpha e^{-i\lambda A} = e^{-\lambda\alpha}|p|^\alpha$.
- (iii) As in(ii), but now, $i[A, r^\alpha] = \alpha r^\alpha$.
- (iv) Follows from (v).
- (v) $\left\|\frac{1}{|p|}\frac{1}{r}\psi\right\|_{L^2} \leq 2\left\|r\frac{1}{r}\psi\right\|_{L^2} = 2\|\psi\|$
 by the uncertainty inequality in 3 dimensions. ■

Proposition 2.4. *Let $n \geq 1$, the hamiltonian H and the momentum operator p are as before. Then, for all n the following estimates hold:*

- a) $E_n^3(H) = E_n(H)E_n(|p|)E_n(H) + E_n(H)\delta^{-1}O(2^{-n})E_n(H)$.
 - b) $E_{\bar{n}}(|p|)E_n(H) = E_{\bar{n}}(|p|)O(2^{-n})E_n(H)$, $\bar{n} \neq n$.
- For $\eta > \ln 2$,
- c) $E_n(|p|)e^{i\eta A}E_n(H) = E_n(|p|)O(2^{-n})E_n(H)$.
 - d) $\bar{E}_n(|p|)E_n(H) = \bar{E}_n(|p|)O(2^{-n})E_n(H)$.

Proof.

Part(a). Let $E_n(H) \equiv E_n$.

$$\begin{aligned} E_n^3(H) &= -E_n(E_n(|p|) - E_n)E_n + E_n(H)E_n(|p|)E_n(H), \\ E_n(|p|) - E_n(H) &= \int \hat{E}_n(\lambda) (e^{i\lambda|p|} - e^{i\lambda H}) d\lambda \\ &= \int \hat{E}_n(\lambda) e^{i\lambda H} \int_0^\lambda e^{-isH} (-i)V e^{is|p|} ds d\lambda. \end{aligned}$$

Therefore, using

$$|V| \leq c < x >^{-2},$$

we get: $|\langle \phi, E_n [E_n(|p|) - E_n(H)] E_n \psi \rangle|$

$$\begin{aligned}
&\leq c \int \left\| \langle x \rangle^{-1} e^{-isH} e^{i\lambda H} E_n \phi \right\| \left| \hat{E}_n(\lambda) \lambda \right| \frac{1}{\lambda} \int_0^\lambda \left\| \langle x \rangle^{-1} e^{-is|p|} E_n \psi \right\| ds d\lambda \\
&\leq c\delta^{-1} \|H E_n \phi\| \int |\lambda \hat{E}_n(\lambda)| d\lambda \| |p| E_n \psi \| \\
&\leq c\delta^{-2} 2^{-n} 2^n 2^{-n} \|\phi\| \|\psi\| = c\delta^{-2} 2^{-n} \|\phi\| \|\psi\|,
\end{aligned}$$

where we used that,

$$\langle x \rangle^{-1} \leq c|p| \leq c\delta^{-1} H$$

in dimension three or higher, and proposition(2.2).

Notice that, $\| |p| E_n \psi \| \leq c\delta^{-1} \|H E_n \psi\| \leq c\delta^{-1} 2^{-n} \|E_n \psi\|$, by Lemma (2.3). Part(b).

$$\begin{aligned}
&|(\phi, E_{\bar{n}}(|p|) E_n(H) \psi)| = |(\phi, E_{\bar{n}}(|p|) [\tilde{E}_{\bar{n}}(|p|) - \tilde{E}_{\bar{n}}(H)] E_n(H) \psi)| \\
&\leq c\delta^{-1} \|H E_n \psi\| \| |p| E_{\bar{n}}(|p|) \phi \| \int |\hat{E}_{\bar{n}} \lambda| d\lambda,
\end{aligned}$$

by the proof of Part(a). The last expression is therefore bounded by,

$$c\delta^{-1} 2^{-n} \|\psi\| 2^{-\bar{n}} \|\phi\| 2^{\bar{n}} = c\delta^{-1} 2^{-n} \|\phi\| \|\psi\| = O(2^{-n}).$$

Part(c) follows from Part(b), since, for $|\lambda| > \ln 2$:

$$\begin{aligned}
E_n(|p|) e^{i\lambda A} E_n &= E_n(|p|) e^{i\lambda A} \tilde{E}_n(p) E_n \\
&+ E_n(p) e^{i\lambda A} \bar{E}_n(p) E_n \\
&= 0 + O(\bar{E}(p) E_n) = O(n 2^{-n}) \text{ from (d).}
\end{aligned}$$

$$\begin{aligned}
\|E_n(|p|) e^{i\lambda A}\| &= \|E_n(|p|) e^{i\lambda A} E_n(e^{-\lambda} |p|)\| \\
&\lesssim \|E_n(|p|) e^{i\lambda A} \sum_{|\bar{n}| \leq M} E_{\bar{n}}(|p|)\| \\
&\text{with } \bar{n} \neq n, M < \infty.
\end{aligned}$$

Part(d) follows from part(b), since the domain of $|p|$, in the support of $\bar{E}_n(|p|)$, is covered by n dyadic intervals from $[2^{-n}, 1]$:

$$\bar{E}_n(p) = 1 - E_n(p) = \sum_{\bar{n} \neq n} E_{\bar{n}}(p) = E(|p| > 1) + \sum_{j=1}^{M \leq n} E_{n_j}.$$

■

Proposition 2.5. *Assume that $H \equiv |p| + V(x)$ has no bound states or zero energy resonances, and that $V(x)$ vanishes faster than r^{-2} at infinity, and is sufficiently regular. The dimension is 3. Then, for some $0 < m < \infty$,*

$$(2.3) \quad H \geq m|p|.$$

Proof. Since H has no bound states, and $V(x) \rightarrow 0$ at infinity, $H \geq 0$. If we now make a small perturbation, $H \rightarrow H_\epsilon = H + \epsilon V$, then, since H has no zero energy resonances, H_ϵ has no bound states, for ϵ sufficiently small. Hence $H_\epsilon \geq 0$. But then,

$$H = \frac{1}{1+\epsilon}(|p| + (1+\epsilon)V) + \left(1 - \frac{1}{1+\epsilon}\right)|p| = \frac{1}{1+\epsilon}H_\epsilon + m|p| \geq m|p|. \blacksquare$$

Lemma 2.6. *For $H = |p| + V$, we have that $|p| \geq \delta H$, for dimension=3, provided,*

$$|V| \leq \bar{c}/r. \text{ Moreover : } E_{I_n}(H)|p|E_{I_n}(H) \geq \delta E_{I_n}(H)HE_{I_n}(H) \geq \delta(\inf I_n) \cdot E_{I_n}(H).$$

Proof. For $\delta > 0$ sufficiently small

$$(2.4) \quad \delta H = \delta|p| + \delta V \leq \delta|p| + \delta c|p| = \delta(1+c)|p| \leq |p|, \delta(1+c) < 1,$$

where we used that for $|V| \leq \bar{c}|x|^{-1}$, we have $|V| \leq c|p|$ in 3 dimensions. So, in particular, we have,

$$(2.5) \quad E_{I_n}(H)|p|E_{I_n}(H) \geq E_{I_n}(H)\delta HE_{I_n}(H) \geq \delta \inf I_n E_{I_n}(H).$$

■

Proposition 2.7. *Suppose, as before, that $H \equiv |p| + V(x)$, the dimension is 3, and that $V(x)$ is sufficiently regular, and vanishes faster than r^{-2} at infinity. Suppose, moreover, that H has no zero energy resonances, and no zero energy bound states. Then,*

$$(2.6) \quad P_c(H)HP_c(H) \geq P_c(H)\delta|p|P_c(H), \quad \text{for some } \delta > 0.$$

Proof. We have that,

$$P_c(H)(|p| + V)P_c(H) \geq 0.$$

Add a small perturbation ϵV to H : $H_\epsilon \equiv H + \epsilon V = |p| + (1+\epsilon)V$. Then, for ϵ sufficiently small, no new bound states are created. Hence,

$$(2.7) \quad f(H_\epsilon \geq -\epsilon_0)(|p| + V + \epsilon V)f(H_\epsilon \geq -\epsilon_0) \geq 0,$$

for some $\epsilon_0 > 0$, and f is a smooth characteristic function of the interval $[-\epsilon_0, \infty]$. Then,

$$P_c(H)HP_c(H) = P_c(H)\left(\frac{1}{1+\epsilon}H_\epsilon + \epsilon_1|p|\right)P_c(H), \epsilon_1 \equiv 1 - \frac{1}{1+\epsilon} = \frac{\epsilon}{1+\epsilon}$$

would imply that:

$$P_c(H)HP_c(H) \geq \epsilon_1 P_c(H)|p|P_c(H),$$

if we can prove that for some $\epsilon_2 < \epsilon_1$,

$$(2.8) \quad P_c(H)H_\epsilon P_c(H) \geq P_c(-\epsilon_2|p|)P_c.$$

So we want to use (2.7) to prove (2.8).

$$(2.9) \quad \begin{aligned} P_c(H)H_\epsilon P_c(H) &= P_c(H)f(H_\epsilon \leq -\epsilon_0)H_\epsilon P_c(H) + P_c(H)f(H_\epsilon \geq -\epsilon_0)H_\epsilon P_c(H) \\ &\geq P_c(H)f(H_\epsilon \leq -\epsilon_0)H_\epsilon P_c(H), \end{aligned}$$

by (2.7). Now,

$$(2.10) \quad \begin{aligned} P_c(H)f(H_\epsilon \leq -\epsilon_0) &= P_c(H)\left[f(H_\epsilon \leq -\epsilon_0) - f(H \leq -\epsilon_0)\right] \\ &= P_c(H) \int \hat{f}(\lambda)e^{+i\lambda H}i \int_0^\lambda e^{-isH}\epsilon V e^{isH_\epsilon} ds d\lambda \\ &= P_c(H)O(\epsilon V) \int |\lambda \hat{f}(\lambda)| d\lambda \\ &= P_c(H)O(\epsilon V)O(\epsilon_0^{-1}), \end{aligned}$$

so,

$$(2.11) \quad \begin{aligned} &P_c(H)f(H_\epsilon \leq -\epsilon_0)H_\epsilon P_c(H) \\ &= P_c(H)O(\epsilon V/\epsilon_0)\tilde{f}(H_\epsilon \leq -\epsilon_0)H_\epsilon \left(\tilde{g}(H_\epsilon \leq -\epsilon_0) + \bar{g}(H_\epsilon \leq -\epsilon_0)\right)P_c(H) \\ &= P_c(H)O(\epsilon V/\epsilon_0)\tilde{f}(H_\epsilon)H_\epsilon \tilde{g}(H_\epsilon \leq -\epsilon_0)P_c(H) \\ &= P_c(H)O(\epsilon V/\epsilon_0)\tilde{f}(H_\epsilon)H_\epsilon O(\epsilon V/\epsilon_0)P_c(H), \end{aligned}$$

since $\tilde{f}(H_\epsilon \leq -\epsilon_0)\bar{g}(H_\epsilon \leq -\epsilon_0) = 0$. ($\tilde{g} + \bar{g} \equiv 1, \tilde{g}\tilde{f} = \tilde{f}$). The last term can be bounded by:

$$(2.12) \quad cP_c(H)\epsilon^2(\epsilon_0^{-2}H_\epsilon)P_c(H) \leq c\epsilon_0^{-2}P_c\epsilon^2|p|P_c(H)(H),$$

$$1/\epsilon_0 \text{ coming from : } \int |\lambda \hat{f}(\lambda)| d\lambda = O(1/\epsilon_0).$$

ϵ_0 is basically the distance of zero to the largest (negative) e.v. of H_ϵ , and ϵ is arbitrarily small, so $\epsilon^2/\epsilon_0^2 \leq O(\epsilon^2)$. Hence, (2.9) - (2.12) imply (2.8). ■

SECTION 3. MAXIMAL VELOCITY BOUND

We begin with estimating, for $a > 1$,

$$(3.1) \quad E_n(H)F_a\left(\frac{|x|}{t} > a\right)E_n(H) \equiv E_nF_aE_n.$$

$$(3.2) \quad E_nF_aE_n = E_nF_a(F_n(A/t) + \bar{F}_n(A/t))E_n,$$

where,

$$(3.3) \quad F_n(A/t) \equiv F_n\left(\frac{A}{t} > R2^{-n}\right) = 1 - \bar{F}_n,$$

$$(3.4) \quad 1 < R < a,$$

and such that

$$(3.5) \quad \bar{F}_n(b \leq R2^{-n}) \equiv \bar{F}_n(b) \quad \text{satisfies} \quad \bar{F}_n(2^{-n}b)F_a(b) = 0.$$

Since, by the propagation estimates,

$$\|F_n(A/t)E_n(H)\psi(t)\| \leq o(1)\|E_n\psi_0\|,$$

with $o(1) \rightarrow 0$ as $t \rightarrow \infty$, we need to control $E_nF_a\bar{F}_nE_n$ by a decaying function of t , in order to prove the maximal velocity bound, on the energy shell I_n .

Proposition 3.1.

$$\begin{aligned} & \|E_n(H)F_a\left(\frac{r}{t}\right)\bar{F}_n(A/t)\| = \\ & \|E_n(H)|p|^{-1}|p|F_a\left(\frac{r}{t}\right)\bar{F}_n(A/t)\| \leq c2^n/t. \end{aligned}$$

Proof.

$$\begin{aligned} E_nF_a\bar{F}_n &= E_n(|p|^{-1}|p|^{-1}\Delta)r^2F_ar^{-2}\bar{F}_n \\ &= E_n|p|^{-1}|p|^{-1}(A^2 + cA + c_1)(r^{-2}F_a)\bar{F}_n \\ &= c_2E_n|p|^{-1}(|p|^{-1}Ar^{-1})(A + c)r^{-1}F_a\bar{F}_n \\ &\quad + c_3E_n|p|^{-1}(|p|^{-1}r^{-1})r^{-1}F_a\bar{F}_n \equiv B_1 + B_2. \end{aligned} \tag{3.6}$$

where in the third equality we used that $[A, r^{-1}] = -ir^{-1}$.

$$(3.7) \quad B_2 = E_n O(2^n) O_0(1) c_3 \frac{1}{at} \left[\left(\frac{ta}{r} \right) F_a \right] \bar{F}_n,$$

where we used that,

$$E_n |p|^{-1} = O(2^{+n}) \quad \text{and} \quad |p|^{-1} r^{-1} = O_0(1).$$

Since $\frac{ta}{r} F_a = O(1)$, $B_2 = O(2^n/at)$,

$$\begin{aligned} B_1 &= E_n O(2^n) O_1(1) c \frac{1}{at} \left(\frac{ta}{r} F_a \right) \bar{F}_n \\ &\quad + E_n O(2^n) O_1(1) \frac{1}{at} \left(\frac{ta}{r} F_a \right) Rt(Rt)^{-1} A \bar{F}_n \\ &\quad + E_n O(2^n) O_1(1) \frac{1}{at} \left[A, \left(\frac{at}{r} F_a \right) \right] \bar{F}_n, \end{aligned}$$

where

$$O_1(1) \equiv |p|^{-1} (A - 3/2i) r^{-1} = |p|^{-1} p \cdot x r^{-1} = \sum_{k=1}^3 \frac{p_k}{|p|} \frac{r_k}{r}.$$

Therefore,

$$\begin{aligned} B_1 &= E_n O(2^n/at) \bar{F}_n E_n + E_n O(2^n) O_1(1) \left(\frac{ta}{r} F_a \right) \frac{R}{a} O(2^{-n}) \bar{F}_n \\ (3.8) \quad &= O(2^n/at) + O\left(\frac{R}{a}\right). \text{ We used } \frac{ta}{r} F_a = O(1), \frac{A}{Rt} \bar{F}_n \lesssim O(2^{-n}). \end{aligned}$$

The key to this computation is the repeated use, as we do below, of the following:

$$\begin{aligned} E_n &= E_n |p|^{-1} |p|^{-1} (A^2 + cA + c_1) r^{-2} \\ &= c_2 E_n |p|^{-1} (|p|^{-1} A r^{-1}) (A + c') r^{-1} + c_3 E_n |p|^{-1} (|p|^{-1} r^{-1}) r^{-1} \\ &= E_n |p|^{-1} O_1(1) (A + c') r^{-1} + c_3 E_n |p|^{-1} (|p|^{-1} r^{-1}) r^{-1} \\ (3.9) \quad &+ c_4 E_n |p|^{-1} (|p|^{-1} r^{-1} A) r^{-1}. \end{aligned}$$

We apply it again, to the $O\left(\frac{R}{a}\right)$ term: $(F^{[k]}(\lambda) \equiv \lambda^k F(\lambda))$;

$$\begin{aligned} O(R/a) &\sim \frac{R}{a} E_n O_1(1) F_a^{[-1]} \bar{F}_n^{[1]} \\ &= \frac{R}{a} E_n |p|^{-1} O_1(1) (A + c') r^{-1} O_1(1) F_a^{[-1]} \bar{F}_n^{[1]} \\ (3.10) \quad &+ \frac{R}{a} E_n |p|^{-1} O_0(1) r^{-1} O_1(1) F_a^{[-1]} \bar{F}_n^{[1]}. \end{aligned}$$

Now, the important observation is that,

$$r^{-1}|p|^{-1}Ar^{-1} = (r^{-1}|p|^{-1}A)r^{-1}$$

and

$$r^{-1}O_1(1) = O_1(1)r^{-1} - r^{-1}O(|p|^{-1})r^{-1} = O_1(1)r^{-1} + O(1)r^{-1}$$

$$(3.11) \quad [A, O_1(1)] \sim O_1(1).$$

We derive:

$$O_1(1)(A+c')r^{-1}O_1(1)F_a^{[-1]}\bar{F}_n^{[1]} = O_1(1)(A+c')(r^{-1}p^{-1}Ar^{-1}+cr^{-1}p^{-1}r^{-1})F_a^{[-1]}\bar{F}_n^{[1]},$$

$$\begin{aligned} Ar^{-1}F_a^{[-1]}\bar{F}_n^{[1]} &= r^{-1}F_a^{[-1]}A\bar{F}_n^{[1]} + cr^{-1}F_a^{[-1]}\bar{F}_n^{[1]} \\ &+ \frac{1}{at}(\partial_r F_a^{[-1]})\bar{F}_n^{[1]} \\ &= \frac{1}{at}F_a^{[-2]}(Rt)2^{-n}\bar{F}_n^{[2]} + \frac{c}{at}F_a^{[-2]}\bar{F}_n^{[1]} + \frac{1}{at}(\partial_r F_n^{[-1]})\bar{F}_a^{[1]} \\ &= O\left(\frac{1}{at}\right) + O(2^{-n}\frac{R}{a}F_a^{[-2]}\bar{F}_n^{[2]}). \end{aligned}$$

$$\partial_\lambda F(\lambda = x) \equiv \left. \frac{\partial F}{\partial \lambda} \right|_{\lambda=x}.$$

Using that $Ar^{-1}p^{-1} = O_1(1) + O_0(1) \equiv O_{0,1}(1)$, we have,

$$\begin{aligned} \frac{R}{a}E_n O_1(1)F_a^{[-1]}\bar{F}_n^{[1]} &= \frac{R}{a}E_n |p|^{-1}O_1(1)(r^{-1}|p|^{-1}A+c')r^{-1}F_a^{[-1]}\bar{F}_n^{[1]} \\ &+ \frac{R}{a}E_n |p|^{-1}O_1(1)^2 r^{-1}F_a^{[-1]}A\bar{F}_n^{[1]} \\ (3.12) \quad &= O\left(\frac{R}{a}\frac{2^n}{at}\right) + E_n \left(\frac{R}{a}\right)^2 O_1(1)^2 F_a^{[-2]}\bar{F}_n^{[2]}. \end{aligned}$$

where we used that, ($L = 0$)

$$E_n |p|^{-1} = E_n O(2^{-n}), \quad r^2 \Delta = A^2 + cA + c',$$

$$\frac{A}{R2^{-n}t}\bar{F}_n = O(1),$$

$$|p|^{-1}Ar^{-1} = O(1).$$

We now do this computation again, this time, for the term of order $(\frac{R}{a})^2$:

Doing it k times, we get:

$$(3.13) \quad \left(\frac{R}{a}\right)^k E_n O(1)^k F_a^{[-k]} \bar{F}_n^{[k]} E_n + \sum_{j=1}^k E_n O\left(\frac{2^n}{t}\right) \left(\frac{R}{a}\right)^{(j-1)} \frac{1}{a} O(1)^j \tilde{F}_a^{[-j]} \bar{F}_n^{[j-1]},$$

with,

$$(3.14) \quad \tilde{F}_a^{[-k]} = \frac{at}{r} F_a^{[-k+1]} + r \partial_r \frac{at}{r} F_a^{[-k+1]}.$$

For $k \sim \delta \ln t$, and $\frac{R}{a} < 1$ sufficiently small, we have that the $E_n F_a \bar{F}_n$ term is bounded by $O(t^{-1} 2^n)$. ■

We can now prove the Theorem on Maximal velocity bound.

Proof of Maximal Velocity Bound

Now,

$$(3.15) \quad \begin{aligned} & \left(\psi(t), F_a^2 \left(\frac{r}{t} > a \right) \psi(t) \right) \\ &= \sum_n \left(H^{-1/2} \psi(t), F_a^2 E_n H^{1/2} \psi(t) \right) + Q \\ &= \sum_n \left(H^{-1/2} \psi(t), F_a^2 (F_n(A/t) + \bar{F}_n(A/t)) E_n H^{1/2} \psi(t) \right) + Q \\ &= \sum_n (H^{-1/2} \psi(t), F_a^2 F_n E_n H^{1/2} \psi(t)) + \sum_n (H^{-1/2} \psi(t), F_a^2 \bar{F}_n E_n H^{1/2} \psi(t)) + Q. \quad \blacksquare \end{aligned}$$

First, we use Proposition 3.1 to control the \bar{F}_n term on the RHS of (3.15)

$$(3.16) \quad \begin{aligned} & |(H^{-1/2} \psi(t), F_a^2 \bar{F}_n(A/t) E_n H^{1/2} \psi(t))| \\ & \leq c \|\langle x \rangle^{1/2} \psi(1)\| \left[\|\bar{F}_n F_a^2 E_n\| \|E_n H^{1/2} \psi(1)\| + \|[F_a^2, \bar{F}_n(A/t)] E_n H^{1/2} \psi(t)\| \right], \end{aligned}$$

where we used that

$$\|H^{-1/2} \psi(t)\| = \|H^{-1/2} \psi(1)\| = \|H^{-1/2} \langle x \rangle^{1/2} \langle x \rangle^{-1/2} \psi(1)\| \leq c \|\langle x \rangle^{1/2} \psi(1)\|.$$

By Proposition 3.1,

$$\|\bar{F}_n(A/t) F_a^2 E_n\| \leq O(2^n/t),$$

hence, the first term in the square bracket of (3.16) is bounded by

$$\begin{aligned} & c2^n/t \|H^{1/2}E_n H^{1/2+\varepsilon} H^{-1/2-\varepsilon} \psi(0)\| \\ & \leq 0(1) \|E_n \psi\| 2^{-n/2} + c2^{-\varepsilon n}/t \| \langle x \rangle^{1/2+\varepsilon} \psi(0) \|. \end{aligned}$$

The second term in the square bracket is

$$\|O\left(\frac{2^n}{t}\right) E_n H^{1/2} \psi(t)\| \leq \frac{c2^n}{t} \|E_n H^{1+\varepsilon} H^{-1/2-\varepsilon} \psi(t)\| \leq \frac{c}{t} 2^{-\varepsilon n} \|\langle x \rangle^{1/2+\varepsilon} \psi(1)\|.$$

Therefore, the sum over n of the \bar{F}_n term in (3.15) is bounded by

$$\left(o(1) + O\left(\frac{1}{t}\right)\right) \|\langle x \rangle^{1/2+\varepsilon} \psi(0)\| \|\langle x \rangle^{1/2} \psi(0)\|, \text{ as } t \rightarrow \infty.$$

$$\begin{aligned} & \left| \sum_n (H^{-1/2} \psi(t), F_a^2 F_n E_n H^{1/2} \psi(t)) \right| \\ & \leq \|F_a H^{-1/2} \psi(t)\| \sum_n \langle n \rangle^{-1/2-\varepsilon/2} \langle n \rangle^{1/2+\varepsilon/2} \|F_n E_n H^{1/2} \psi(t)\| \\ & \leq c \|F_a H^{-1/2} \psi(t)\| \left(\sum_n \langle n \rangle^{1+\varepsilon} \|F_n E_n H^{1/2} \psi(t)\|^2 \right)^{1/2} \\ & \leq c \|F_a H^{-1/2} \psi(t)\| \left(\sum_n o(1) \|H^{1/2} E_n \langle x \rangle^{1/2} \psi(1)\|^2 \langle n \rangle^{1+\varepsilon} \right)^{1/2} \\ (3.17) \quad & \leq c \|F_a H^{-1/2} \psi(t)\| o(1) \|\langle x \rangle^{1/2} \psi(1)\|, \end{aligned}$$

$$\begin{aligned} & \left| \sum_n (H^{-1/2} \psi(t), F_a^2 \bar{F}_n E_n H^{1/2} \psi(t)) \right| \leq \left| \sum_n (F_a H^{-1/2} \psi(t), \bar{F}_n F_a E_n H^{1/2} \psi(t)) \right| \\ & \quad + \left| \sum_n (F_a H^{-1/2} \psi(t), O\left(\frac{1}{at}\right) (R2^{-n})^{-1} O(1) E_n H^{1/2} \psi(t)) \right| \\ (3.18) \quad & \leq c \left\| F_a H^{-1/2} \psi(t) \right\| \sum_n \left\| 2^{+n/2} E_n \psi(t) \right\| \frac{1}{t} \frac{R}{a}. \end{aligned}$$

■

It follows that,

$$(3.19) \quad \begin{aligned} (\psi(t), F_a^2 \psi(t)) &\leq c \left\| F_a H^{-1/2} \psi(t) \right\| \left\| |\ln H|^{\frac{1+\varepsilon}{2}} H^{-1/2} < x >^{1/2} \psi_0 \right\| o(1) + Q, \\ \|F_a H^{-1/2} \psi(t)\| &\leq \|H^{-1/2} \psi(t)\| = \|H^{-1/2} \psi(0)\| \leq c \|< x >^{1/2} \psi(0)\| \end{aligned}$$

$$Q \equiv \left(H^{-1/2} \psi(t), \left[H^{1/2}, F_a^2 \right] \psi(t) \right).$$

To control Q , we need to commute fractional powers of H . To this end we use that:

$$H^\alpha P_c(H) = c_\alpha \int_0^\infty \frac{\lambda^{\alpha-1}}{\lambda + H} H P_c(H) d\lambda,$$

and estimate,

$$(3.20) \quad \begin{aligned} \lambda^{\alpha-1} \left[\frac{H}{H + \lambda}, F_a \right] &= \lambda^{\alpha-1} H \left[\frac{1}{H + \lambda}, F_a \right] + [H, F_a] \frac{\lambda^{\alpha-1}}{H + \lambda} \\ &= \frac{-H}{H + \lambda} O\left(\frac{1}{t} F'_a\right) \frac{\lambda^{\alpha-1}}{H + \lambda} + O\left(\frac{1}{t} F'_a\right) \frac{\lambda^{\alpha-1}}{H + \lambda} \equiv \circledast \\ \int_0^\infty d\lambda \circledast &= O(1) O\left(\frac{1}{t} F'_a\right) O(H^{\alpha-1}). \end{aligned}$$

Therefore, using (3.20) with $\alpha = 1/2$, we have that

$$(3.21) \quad |Q| = |(H^{-1/2} \psi(t), O(1) O(\frac{1}{t}) H^{-1/2} \psi(t))| \leq \frac{c}{t} \|H^{-1/2} \psi(t)\|^2 \leq \frac{c}{t} \|< x >^{1/2} \psi(0)\|^2. \blacksquare$$

End of Proof.

SECTION 4. MINIMAL VELOCITY BOUNDS

We will use the analysis developed so far to obtain lower bounds on the speed of propagation. For this, we consider the family of observables given by

$$(4.1) \quad \Phi_n \equiv E_{I_n} F_n \left(\frac{A}{t R 2^{-n}} \geq 0 \right) E_{I_n}, \quad 0 \leq R < 1.$$

$$E_{I_n} = E_{I_n}(H) = E(H \in I_n), \quad n \geq 0.$$

Equation (0.12) applies, and we get

$$(4.2) \quad \partial_t \langle \psi(t), \Phi_n \psi(t) \rangle =$$

$$(4.2a) \quad = \langle \psi(t), \left\{ i[H, \Phi_n] + \frac{\partial \Phi_n}{\partial t} \right\} \psi(t) \rangle$$

$$(4.2b) \quad = \frac{1}{t} \langle E_n \psi(t), \left[\frac{-A}{R2^{-n}} F'_n \left(\frac{A}{tR2^{-n}} \right) + |p|^{1/2} F'_n(A/t) |p|^{1/2} \frac{1}{R2^{-n}} \right.$$

$$(4.2c) \quad \left. + 2Re|p|^{1/2} R_2(A/t) + i[tV, F_n] \right] E_n \psi(t) \rangle$$

$$(4.2d) \quad = \frac{1}{t} \langle \psi(t), E_n \left\{ u F'_n(u) |_{u=A/Rt2^{-n}} + |p|^{1/2} F'_n |p|^{1/2} \frac{1}{R2^{-n}} \right\} \psi(t) \rangle$$

$$(4.2e) \quad + \frac{1}{t} \langle E_n \psi(t), 2Re|p|^{1/2} R_2(A/t) E_n \psi(t) \rangle + \langle E_n \psi(t), i[V, F_n] E_n \psi(t) \rangle \equiv B_{ps} + B_{re} + B_V.$$

Theorem 4.1. *We assume the previous notation, and the hamiltonian H is as before. Then, for all $n > N(\|V\|)$,*

$$(4.3) \quad \int_1^\infty \|\tilde{F}_n(A/t) E_n(H) \psi(t)\|^2 \frac{dt}{t} < C < \infty$$

For

$$(4.4) \quad \tilde{F}_n(u) = F(|u| < (1 - \varepsilon)2^{-n})$$

Proof. The result will follow by integration from equation (4.2) if we can show that:

(a)

$$B_{ps} \geq \frac{c}{t} \langle E_n \psi(t), \tilde{F}_n^2(A/t) E_n \psi(t) \rangle + B_{ps}^1(t), c > 0;$$

for all n large enough, depending only on norms of V .

$$(4.5) \quad B_{ps}^1(t) \in L^1(dt) : \int_1^\infty B_{ps}^1(t) dt \leq C \|E_n \psi(0)\|^2,$$

(b)

$$(4.6) \quad \int_1^T B_{re} dt \leq c \|E_n \psi(0)\|^2$$

(c)

$$(4.7) \quad B_V \leq \delta_0 B_{ps} + B_V^1(t), \delta_0 \leq 1/2, \text{ and } \int_1^T B_V^1(t) dt \leq c \|E_n \psi(0)\|^2.$$

First, we prove assertion (a).

Since $F'_n(A/t)$ is a bump function localizing

$$\frac{A}{tR2^{-n}} \sim 0,$$

it follows that the first term of (4.2b) is bounded below by

$$(4.8) \quad \frac{-A}{tR2^{-n}} F'_n \geq -\delta \tilde{F}_n^2 \left(\frac{A}{tR2^{-n}} \sim 0 \right).$$

δ is a small number depending on the sharpness η of the function

$$(4.9) \quad \begin{aligned} & \tilde{F}(x \sim 0)^2 \equiv F'(x \geq 0); \\ & F \in C^\infty(\mathbb{R}), \quad F_0(x \geq 0) = \begin{cases} \frac{1}{2\eta} + x/\eta & |x| < \eta/2 \\ 0 & x \leq -\eta/2 \\ 1 & x \geq \eta/2 \end{cases} \end{aligned}$$

$$(4.10) \quad F \equiv g_{\eta/10} * F_0$$

where $g_{\eta/10}$ is an approximate δ -function with support size $\eta/10$.

The second term that contributes to (4.2b) is

$$(4.11) \quad \frac{1}{t} \frac{2^n}{R} \langle \psi(t), E_n |p|^{1/2} F'_n(A/t) |p|^{1/2} E_n \psi(t) \rangle =$$

$$(4.12) \quad \frac{1}{t} \frac{2^n}{R} \langle \psi(t), \tilde{F}_n E_n |p| E_n \tilde{F}_n \psi(t) \rangle$$

$$\begin{aligned}
& + \frac{1}{t} \frac{2^n}{R} \left\{ \langle \psi(t), E_n \left[[E_n |p|^{1/2}, \tilde{F}_n] \tilde{F}_n |p|^{1/2} + \tilde{F}_n E_n |p|^{1/2} [\tilde{F}_n, E_n |p|^{1/2}] \right] E_n \psi(t) \rangle \right. \\
& \qquad \qquad \qquad \geq \frac{1}{t} \frac{1}{R} \langle \psi(t), \tilde{F}_n E_n E_n \tilde{F}_n \psi(t) \rangle \\
(4.13) \quad & \left. + \frac{1}{t} \frac{2^n}{R} \langle \psi(t), E_n \left(C_n \tilde{F}_n |p|^{1/2} - \tilde{F}_n E_n |p|^{1/2} C_n \right) E_n \psi(t) \rangle \right\}
\end{aligned}$$

with

$$(4.14) \qquad C_n \equiv [E_n |p|^{1/2}, \tilde{F}_n].$$

We used in the above the following *Quantitative Mourre Estimate*:

$$\begin{aligned}
E_n(H) |p| E_n(H) &= E_n(H) H E_n(H) - E_n(H) V E_n(H) \geq E_n(H) (2^{-n} - c \langle x \rangle^{-1-\varepsilon}) E_n(H) \\
&\geq (1 - \varepsilon) E_n(H) 2^{-n} E_n(H)
\end{aligned}$$

since $\|E_n(H)\| < \langle x \rangle^{-1-\varepsilon} \leq 2^{-n(1+\varepsilon)}$.

Next, we need the following Proposition, showing that C_n is higher order correction (in t).

Proposition 4.2. *For \tilde{F}_n defined as above, we have*

$$(4.15) \qquad [E_n(H) |p|^{1/2}, \tilde{F}_n(A/t)] = O\left(\frac{1}{t}\right) 2^{n/2}$$

b)

$$(4.16) \quad i[E_n(H), A] = H E_n'(H) + \tilde{E}_n(H) O(\|\langle x \rangle^{-\sigma} \tilde{E}_n(H)\|) O\left(\int |\lambda \hat{E}_n(\lambda)|\right) = O(1)$$

c)

$$(4.17) \qquad [E_n(H), \tilde{F}_n(A/t)] = O(2^n/t)$$

The proof of this Proposition is postponed to the end of the proof of theorem. For $t \leq K2^n$, we have that $(\psi(t) = E_n \psi(t))$

$$\begin{aligned}
& \partial_t \langle \psi(t), \Phi(t) \Phi(t) \rangle - \langle \psi(t), i[H, \Phi(t)] \psi(t) \rangle \\
&= \langle \psi(t), \frac{\partial \Phi}{\partial t} \psi(t) \rangle \\
(4.18) \quad & \sim -\langle \psi(t), \delta \tilde{F}_n^2 \left(\frac{A}{Rt2^{-n}} \sim 0 \right) \psi(t) \rangle
\end{aligned}$$

Integrating on the interval $1 \leq t \leq 2^n$, we have that

$$\begin{aligned}
& \delta \int_1^t \langle \psi(t), \tilde{F}_n^2(A/t) \psi(t) \rangle dt \\
& + \langle \psi(T), \Phi(T) \psi(T) \rangle - \langle \psi(1), \Phi(1) \psi(1) \rangle \\
& \leq \int_1^T |\langle \psi(t) (HE_n \Phi - \Phi E_n H) \psi(t) \rangle| dt \\
(4.19) \quad & \leq \int_1^{K2^n} \|E_n \psi(t)\|^2 \|HE_n\| dt \leq CK \|E_n \psi(0)\|^2.
\end{aligned}$$

For $t \geq K2^n$, we need to bound the second term in the expression (4.13) by an $L^1(dt)$.

By using the proposition, part (a) it follows that

$$\begin{aligned}
(4.13) & \geq \frac{1}{Rt} \langle \psi(t), \tilde{F}_n E_n \tilde{F}_n \psi(t) \rangle \\
(4.20) \quad & - 2 \frac{2^n}{Rt} \|E_n \psi(t)\|^2 \|C_n\| \| |p|^{1/2} E_n \| \\
& \geq -C \frac{2^n}{(Rt)^2} \|E_n \psi(t)\|^2 + \frac{1}{Rt} \langle \psi(t), \tilde{F}_n E_n \tilde{F}_n \psi(t) \rangle.
\end{aligned}$$

The integral of first term on the RHS of (4.20) is bounded by

$$(4.21) \quad \int_{K2^n}^\infty C \frac{2^n}{(Rt)^2} \|E_n \psi(t)\|^2 dt \leq \frac{C}{KR^2} \|E_n \psi(0)\|^2.$$

Using part (c) of Proposition (4.2), we get that

$$\begin{aligned}
(4.22) \quad \tilde{F}_n E_n \tilde{F}_n &= E_n \tilde{F}_n^2 + [\tilde{F}_n, E_n] \tilde{F}_n \\
&= E_n \tilde{F}_n^2 + O(2^n/t) \tilde{F}_n.
\end{aligned}$$

Therefore

$$(4.23) \quad \begin{aligned} \frac{1}{Rt} \langle \psi(t), \tilde{F}_n E_n \tilde{F}_n \psi(t) \rangle &= \frac{1}{Rt} \langle E_n \psi(t), \tilde{F}_n^2 E_n \psi(t) \rangle \\ &+ \frac{2^n}{Rt^2} \langle \psi(t), O(1) \tilde{F}_n E_n \psi(t) \rangle \end{aligned}$$

The L^1 norm $L^1(K2^n, \infty)$ of the second term of the RHS of equation (4.23) is bounded by $\frac{C}{RK} \|E_n \psi(0)\|^2$ as in (4.21).

This proves assertion (a).

Assertion (b) is the integrability, up to small corrections of the $R_2(A/t)$ term, and it follows from the proof of the maximal velocity estimate, Proposition (3.1)

Assertion (c) is control of the potential term, by an $L^1(dt)$ part and the first term, the first term being

$$\frac{\delta}{t} \langle \psi(t), E_n \tilde{F}_n(A/t) E_n \psi(t) \rangle$$

First, we rewrite the potential term.

$$(4.24) \quad \begin{aligned} &\langle \psi(t), i[H, E_n F_n E_n] \psi(t) \rangle \\ &= \langle \psi(t), iE_n [H, F_n] E_n \psi(t) \rangle \end{aligned}$$

So, the potential term is

$$(4.25) \quad \begin{aligned} V_n &\equiv \langle \psi(t), iE_n [V, F_n] E_n \psi(t) \rangle \\ &= \langle \psi(t), iE_n (\tilde{E}_n V F_n - F_n V \tilde{E}_n) E_n \psi(t) \rangle \\ &\text{with } E_n \tilde{E}_n = E_n. \end{aligned}$$

Then, commute another \tilde{E}_n through the F_n .

$$(4.26) \quad \begin{aligned} V_n &= \langle \psi(t), iE_n (\tilde{E}_n V \tilde{E}_n F_n + \tilde{E}_n V [F_n, \tilde{E}_n] \\ &\quad - F_n \tilde{E}_n V \tilde{E}_n - [\tilde{E}_n, F_n] V \tilde{E}_n) E_n \psi(t) \rangle \\ &= \langle \psi(t), iE_n \left\{ [\tilde{E}_n V \tilde{E}_n, F_n] + \tilde{E}_n V [F_n, \tilde{E}_n] \right. \\ &\quad \left. - [\tilde{E}_n, F_n] V \tilde{E}_n \right\} E_n \psi(t) \rangle \end{aligned}$$

Since by Proposition (4.2) part (c)

$$(4.27) \quad \|[\tilde{E}_n, F_n(A/t)]\| \leq C2^n/t$$

$$(4.28) \quad \text{and } \|V\tilde{E}_n\| \leq C2^{-n(1+\varepsilon)} \text{ for}$$

$$(4.29) \quad |V| = O(\langle x \rangle^{-\sigma}), \sigma > 1.$$

These terms decay like $2^{-\varepsilon n}t^{-1}$.

It remains to estimate $[\tilde{E}_n V \tilde{E}_n, F_n(A/t)]$.

$$(4.30) \quad \begin{aligned} i[A, \tilde{E}_n(H)V\tilde{E}_n(H)] &= \\ &= i[A, \tilde{E}_n(H)]V\tilde{E}_n(H) + \tilde{E}_n i[A, V]\tilde{E}_n + \tilde{E}_n V i[A, \tilde{E}_n(H)] \\ &= \tilde{E}_n O(\langle x \rangle^{-\sigma})\tilde{E}_n + O(1)O(\langle x \rangle^{-\sigma})\tilde{E}_n + \tilde{E}_n O(\langle x \rangle^{-\sigma})O(1) \\ &= O(2^{-n(1+\varepsilon)}) \end{aligned}$$

for $\sigma > 1$.

ε can be taken to be $\frac{1}{2} - 0$, provided $\sigma \geq 3/2$.

Using (4.30), we obtain

$$\begin{aligned} \|F_n(A/t), \tilde{E}_n(H)V\tilde{E}_n(H)\| &= \\ \int \hat{F}_n(\lambda)e^{i\lambda A/t} \int_0^\lambda e^{-isA/t} \frac{1}{t} [A, \tilde{E}_n(H)V\tilde{E}_n(H)], e^{+isA/t} ds d\lambda \| \\ &\leq c_V \int |\hat{F}_n(\lambda)\lambda| d\lambda 2^{-n(1+\varepsilon)}/t \\ &\leq c_V(2^n/Rt)2^{-n(1+\varepsilon)} = c_V 2^{-\varepsilon n}/Rt. \end{aligned}$$

So, the potential term is higher order, and the proof is complete.

■

Proof of Proposition (4.2)

Part (a)

$$E_n(H)|p|^{1/2} = O(2^{-n/2}).$$

Therefore

$$\begin{aligned} [E_n(H)|p|^{1/2}, \tilde{F}_n(A/t)] &= \\ i \int \hat{\tilde{F}}_n(\lambda)e^{i\lambda A/t} \int_0^\lambda e^{-isA/t} \frac{1}{t} [A, E_n(H)|p|^{1/2}] e^{-isA/t} ds d\lambda \\ [A, E_n(H)|p|^{1/2}] &= [A, E_n(H)]|p|^{1/2} + E_n(H)[A, |p|^{1/2}] \end{aligned}$$

$$\begin{aligned}
&= [HE'_n(H)\tilde{E}_n(H) + O(2^n)\|\langle x \rangle^{-\sigma}\tilde{E}_n(H)H^{1/2}\| \|H^{-1/2}|p|^{1/2}\|] \tilde{E}_n|p|^{1/2} + \frac{i}{2}E_n(H)|p|^{1/2} \\
&= O(2^{-n/2}) + \|\langle x \rangle^{-\sigma}\tilde{E}_n(H)\| O(2^{-n/2}) + O(2^{-n/2}) = \\
&\quad O(2^{-n/2}),
\end{aligned}$$

by part (b) and for $\sigma \geq 1$.

Therefore,

$$\begin{aligned}
&\|[E_n(H)|p|^{1/2}, \tilde{F}_n(A/t)]\| \\
&\leq \int |\lambda \hat{\tilde{F}}_n(\lambda)| d\lambda \frac{1}{t} 2^{-n/2} = O(2^{n/2}/t).
\end{aligned}$$

Part (b)

$$\begin{aligned}
[E_n, A] &= [E_n\tilde{E}_n, A] = [E_n, A]\tilde{E}_n + [E_n, A]\tilde{E}_n \\
[E_n(H), A]\tilde{E}_n &= \int \hat{E}_n(\lambda) e^{i\lambda H} \int_0^\lambda e^{-isH} [H, A] e^{isH} ds d\lambda \tilde{E}_n \\
&= \int \hat{E}_n \lambda e^{i\lambda H} \int_0^\lambda e^{-isH} (-i)[H - V - X \cdot \nabla V] \tilde{E}_n e^{isH} ds d\lambda \\
&= -iHE'_n(H)\tilde{E}_n - i \int \hat{E}_n(\lambda) e^{i\lambda H} \int_0^\lambda e^{-isH} O(\langle x \rangle^{-\sigma}) \tilde{E}_n e^{isH} ds d\lambda \\
&= -iHE'_n(H)\tilde{E}_n + O\left(\int |\lambda \hat{E}_n(\lambda)| \langle x \rangle^{-\sigma} \|\tilde{E}_n\|\right)
\end{aligned}$$

The last term is bounded by

$$\int |\lambda \hat{E}_n(\lambda)| d\lambda \|\langle x \rangle^{-\sigma} \tilde{E}_n(H)\| = O(2^n) O(2^{-\sigma n}), \sigma \leq 3/2.$$

which completes the proof of part (b).

Part (c)

$$\begin{aligned}
&[E_n(H), \tilde{F}_n(A/t)] = \\
&\int \hat{\tilde{F}}_n(\lambda) e^{i\lambda A/t} \int_0^\lambda e^{-isH/t} [A/t, E_n(H)] e^{isA/t} ds d\lambda.
\end{aligned}$$

Taking the norm, it is bounded by

$$\int |\lambda \hat{\tilde{F}}_n(\lambda)| \frac{1}{t} \|[A, E_n(H)]\| = O\left(\frac{2^n}{t}\right) O(1).$$

by part (b).

■

Next, we need the following estimate, similar to Proposition (3.1):

Localization lemma.

$$(4.31) \quad \|E_n(H)F_b\left(\frac{|x|}{t} < b\right)F_n\left(\frac{A}{2^{-n}Rt} > 1\right)E_n(H)\| = O\left(\frac{2^n}{t}\right)$$

for t large, provided $\frac{b}{R} < 1$.

Proof.

Denoting momentarily $E_n(H) \equiv E_n$,

$$F_b\left(\frac{|x|}{t} < b\right) \equiv F_b \text{ and } F_n\left(\frac{A}{2^{-n}Rt} > 1\right) \equiv F_n,$$

we have, using that $A = \frac{1}{2}(x \cdot p + p \cdot x) = \sum_{j=1}^3 x_j p_j + c$:

$$\begin{aligned} E_n F_b F_n E_n &= E_n F_b x_j (F_n A^{-1}) p_j E_n \\ &\quad + c E_n F_b (F_n A^{-1}) E_n \\ &+ \frac{2^n}{Rt} E_n F_b x_j [p_j, \tilde{F}_n] E_n \equiv E_n F_b x_j (F_n A^{-1}) p_j E_n + I_1 + I_2 = \\ &= \frac{b}{R} E_n \tilde{F}_b \tilde{F}_n (E_n H) H^{-1} \cdot p 2^n \\ &+ \frac{b}{R} E_n \tilde{F}_b \tilde{F}_n \cdot [p, E_n] 2^n + I_1 + I_2 \\ &\equiv J_1 + J_2 + I_1 + I_2 \end{aligned}$$

where we define

$$\begin{aligned} \tilde{F}_b &= \frac{x}{bt} F_b = (x_1, x_2, x_3) \frac{1}{bt} F_b \\ \tilde{F}_n &= (Rt 2^{-n}) A^{-1} F_n \end{aligned}$$

and summation over j is implied; \cdot stands for scalar product (\tilde{F}_b and p are vectors).

We then have

$$\|J_1\| \lesssim \frac{b}{R} \|E_n \tilde{F}_b \tilde{F}_n E_n\| 2^{-n} 2^n = \frac{b}{R} \|E_n \tilde{F}_b \tilde{F}_n E_n\|$$

from $\|H^{-1}|p|\| \lesssim 1$.

$$\|J_2\| \leq \frac{b}{R} \|E_n \tilde{F}_b \tilde{F}_n\| 2^n \|[p, E_n(H)]\| \leq \frac{b}{R} 2^{-n/2+\varepsilon n} \|E_n \tilde{F}_b \tilde{F}_n\|$$

from $\|[p, E_n(H)]\| < 2^{-(\frac{3}{2}-\varepsilon)n}$

$$\begin{aligned}
I_1 &= c \frac{2^n}{Rt} E_n F_b \tilde{F}_n E_n \\
I_2 &= \frac{b}{R} E_n \tilde{F}_b \left(\frac{2^n}{Rt} \right) \cdot 2^n O(\tilde{F}'_n) p E_n \\
&\approx \frac{b}{R} \frac{2^n}{Rt} E_n \tilde{F}_b O(\tilde{F}'_n) E_n
\end{aligned}$$

So, I_1 and I_2 are higher order, of the form

$$\begin{aligned}
&\frac{2^n}{Rt} E_n O(1) E_n \\
J_2 &\lesssim E_n \tilde{F}_b \tilde{F}_n O\left(\frac{b}{R} 2^{-\frac{n}{2} + \varepsilon n}\right)
\end{aligned}$$

So, J_1 is the leading term.

$$\begin{aligned}
|\tilde{F}_b| &= \left| \frac{x}{bt} F\left(\frac{|x|}{t} < b\right) \right| \leq F_b\left(\frac{|x|}{t} < b\right) \\
\tilde{F}_n &= A^{-1} F_n(A/t) (tR2^{-n}) \leq F_n
\end{aligned}$$

So

$$\begin{aligned}
\|J_1\| &\leq \frac{b}{R} \|E_n F_b \frac{\tilde{F}_b}{F_b} F_n \frac{\tilde{F}_n}{F_n} E_n\| \\
&\leq \frac{b}{R} \|E_n F_b F_n E_n\| \\
&\quad + \frac{b}{R} \| [E_n, \tilde{F}_b/F_b] F_b \tilde{F}_n E_n \| \\
&\quad + \frac{b}{R} \| E_n F_b F_n [\tilde{F}_n/F_n, E_n] \| \\
G_n &\equiv \frac{\tilde{F}_n}{F_n} = F(A > Rt2^{-n}) / (A / (Rt2^{-n})) \\
G_b &\equiv \frac{\tilde{F}_b}{F_b}
\end{aligned}$$

and the proof follows since $[F_b, E_n] = O(2^n/t) = [G_b, E_n]$, and

$$[F_b, E_n] \text{ and } [F_n, E_n] = O(2^n/t).$$

■
The above analysis is now used to prove minimal velocity bounds, following similar arguments leading to the maximal velocity bounds:

$$(4.32) \quad \langle \psi(t), F_b(\frac{|x|}{t} < b) \psi(t) \rangle = \langle H^{-1/2} \psi(t), F_b(\frac{|x|}{t} < b) H^{1/2} \psi(t) \rangle + Q(t).$$

$Q(t)$ comes from commuting $H^{1/2}$ through F_b . As shown before, this term goes to zero as time tends to infinity, as in equation (4.21). So, we only need to estimate the first term of equation (4.32).

$$(4.33) \quad \sum_n \langle H^{-1/2} \psi(t), F_b(\frac{|x|}{t} < b) E_n^2 H^{1/2} \psi(t) \rangle = \sum_n [\langle H^{-1/2} \psi(t), E_n F_b(\frac{|x|}{t} < b) E_n H^{1/2} \psi(t) \rangle + \langle H^{-1/2} \psi(t), O(2^n/t) E_n H^{1/2} \psi(t) \rangle]$$

As before, the second term of (4.33) is controlled by

$$\frac{c}{t} \| \langle x \rangle^{1/2} \psi(0) \|^2.$$

So, it remains to control the first part of (4.33).

Proceeding as before, we write in this first term

$$E_n F_b(\frac{|x|}{t}) E_n = E_n F_b(\frac{|x|}{t} < b) (F_n(\frac{A}{2^{-n} R t}) + \bar{F}_n(\frac{A}{2^{-n} R t})) E_n.$$

We have shown in the localization lemma that

$$E_n F_b F_n E_n = O(2^n/t).$$

The propagation estimates of section 4 imply:

$$\| E_n F_b F_n E_n H^{1/2} \psi(t) \| = o(1) 2^{-n} \| \langle x \rangle^{1/2} \psi(0) \|_2.$$

$$\| H^{-1/2} \psi(t) \| = \| H^{-1/2} \psi(0) \| \lesssim \| \langle x \rangle^{1/2} \psi(0) \|_2.$$

Putting it all together, this term also vanishes as time goes to infinity. So, we finally have:

Theorem-Minimal Velocity Bound. *For H as before, we have for $b < 1$:*

$$\begin{aligned} \langle \psi(t), F_b(\frac{|x|}{t} < b) \psi(t) \rangle &= o(1) \| \langle x \rangle^{1/2} \psi(0) \|_2^2 \\ \int_1^\infty \langle \psi(t), F_b(\frac{|x|}{t} < b) \psi(t) \rangle \frac{dt}{t} &\leq c \| \langle x \rangle^{1/2} \psi(0) \|_2^2. \end{aligned}$$

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